

FURTHER DEVELOPMENT OF AUTOGYRO THEORY.

 By C. N. H. Lock, M.A.

 PART I.

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Summary. Introductory.—The general theory of the autogyro given by Glauert in R. & M. 1111 is based on certain simplifying approximations and assumptions. The object of the present paper is to develop the theory still further by removing some of the approximations.

The approximations of R. & M. 1111 may be classified as follows :—

- (1) The coefficient of axial velocity through the disc is constant over the disc and is a small quantity.
- (2) The lift coefficient of a blade element is proportional to the incidence, and the profile drag coefficient is constant.
- (3) The flapping motion is expanded as a Fourier's series and coefficients of $\cos 2 \psi$, $\sin 2 \psi$ etc. are neglected.
- (4) Squares and higher powers of the ratio of the forward speed to the tip speed ($\mu = V \cos i/R \Omega$) are neglected throughout.

Range of Investigations. Part I.—Assumption 4 is dispensed with, all powers of μ being retained. It had been remarked that it is theoretically possible to eliminate the flapping motion of an autogyro by substituting a suitable mechanical variation of the blade angle round the circle. According to the formulae of R. & M. 1111, the modified machine would have about double the maximum lift/drag in a standard case. It was argued that the modified machine is in fact equivalent to the normal machine if the motion of the blades is referred to the plane in which the blades move, which is not normal to the axle.

The present investigation verifies that the two machines are, in fact, identical. It appears that whereas the longitudinal force is correctly estimated in R. & M. 1111 (to the first order), the possible error in the estimation of the incidence due to neglect of terms of order μ^2 , is of such a magnitude as to allow an error in the contribution of the lift to the drag for given lift, of the same order of magnitude as the contribution to the drag of the whole of the longitudinal force. This explains the fictitious discrepancy between the values of maximum lift/drag for the two machines.

After working out the force components, thrust and longitudinal force for both cases and verifying their identity, an alternative method of determining the drag is developed, based on considerations of energy loss; it has been verified that the resulting formulae give results identical with those already obtained, and, being more simple, take the place of the rather complicated formulae for the longitudinal force.

The ratio of the value of maximum lift/drag of the present investigations (the true value for heavy blades subject to assumptions 1 and 2) to the value given in R. & M. 1111 is as follows:—

Blade angle	..	0°	2°	4°
ratio	1.025	1.44	2.1

On account of the errors introduced by assumptions 1, 2 and 3 (especially 2), the actual value of lift/drag is probably lower than that obtained here, and may be in fact closer to the value obtained in R. & M. 1111.

Part II takes account of the general term in the flapping motion so as to remove the restriction to infinitely heavy blades.

Further development.—It is proposed to replace the constant profile drag coefficient and the lift curve of uniform slope (Assumption 2) by the actual values obtained in aerofoil tests, so as to take account of the stalling of the blade sections. It has been found possible (on certain approximations to the flapping motion), to replace the double graphical integration over the airscrew disc by a single graphical integration, in which the variable is a simple function of the incidence of a section.

1. *General Introduction.*—The autogyro is essentially a windmill of low pitch working in a sidewind, and it is natural to apply to it the modern methods of strip theory combined with the Prandtl theory of interference, which have been so successful in the case of the monoplane wing and the ordinary airscrew. The autogyro problem presents special difficulties; the sidewind velocity is comparable with the tip speed in an important part of the working range, while the flapping motion of the blades is of primary importance.

In R. & M. 1111* Glauert has developed a comprehensive theory of the autogyro based on certain simplifying approximations. The present report is an attempt to carry the theory a stage further by removing some of these approximations.

(1) *The axial component velocity u is assumed to be constant over the autogyro disc.† It differs from $V \sin i$, the component of the autogyro velocity in the same direction, by an amount calculable by the Prandtl formula for the interference of a monoplane wing.* The value of u is, however, always small in comparison with the tip velocity $R \Omega$, and squares of the ratio of $u/R \Omega$ are neglected throughout. The actual value of $u/R \Omega$ varies from 0.02 to 0.05 in practice and the last assumption is therefore thoroughly justified. The first assumption is more doubtful but it may be noticed that when the forward speed of the machine is greatest and the percentage variation of the interference velocity v over the disc is likely to be most important, then $v (= V \sin i - u)$ is least.

* R. & M. 1111. A general theory of the autogyro. By H. Glauert, M.A.

† Actually the effect of a variation of u of special type is treated in section 10 of R. & M. 1111.

(2) It follows from the smallness of the ratio $u/R\Omega$ ($= x$) and the smallness of the blade angle θ , that the angle of incidence of the blade section is in general small over the greater part of the airscrew disc. This is the justification for the assumptions made as to the nature of the aerodynamic forces. *The resultant force is assumed to lie in a plane normal to the blade and to depend only on the component velocity in this plane and not on the radial velocity. The lift coefficient is assumed to be proportional to the angle of incidence, since the aerofoil sections are of symmetrical shape. Also, since the profile drag contributes only a small correction to the force components due to the lift, the actual drag coefficients are replaced by a mean value δ .*

(3) The effect of the flapping motion is discussed by expanding the flapping angle β , as a Fourier's series in ψ , the angular displacement of the autogyro at any instant. *The first three terms (constant, coefficient of $\cos \psi$, coefficient of $\sin \psi$) are retained, all other terms being neglected.*

(4) It is convenient to write:—

$$\text{component velocity parallel to axle} = u = x R\Omega^*,$$

(as in R. & M. 1111).

$$\text{component velocity normal to axle} = \mu R\Omega.$$

Actually the interference on the velocity component normal to the axle is assumed to be negligible so that $\mu R\Omega = V \cos i$. *In obtaining the expressions (on page 564, R. & M. 1111) for the flow at a blade element, squares and higher powers of μ are neglected throughout.*

The results of the present report are as follows.

Part I.—Assumption 4 is dispensed with, all terms in μ being retained. The importance of this increases with the speed of the autogyro. At the maximum flying speed of the existing machine μ has the value 0.4, and it is desirable to extend the theory to values of μ which are as large as possible. In R. & M. 1111 Glauert considers that his approximations are valid for values of μ not greater than 0.5.

In R. & M. 1111 the forces on the whole autogyro are obtained in the form of the components:—Thrust T parallel to the axle and longitudinal force H normal to the axle. The drag of the machine is expressed in the form—

$$D = T \sin i + H \cos i.$$

It appears on examination of the results of R. & M. 1111 that while H is correctly estimated to the first order in μ , the neglect of terms of order μ^2 compared with those retained in the expression for zero torque involves an error in the determination of i which affects the

* A list of symbols is given in Appendix I., p. 616.

drag to the same order as the whole contribution of the longitudinal force. The importance of the longitudinal force rises with increase of forward speed and of μ ; hence, the present extension of the theory of R. & M. 1111 becomes important at the highest forward speeds and in particular in determining the maximum value of lift/drag. The error is also of the same order as the whole effect of the freedom of the blades to flap. On the other hand, the error becomes small at large angles of incidence and so does not affect the results of R. & M. 1111 in the neighbourhood of maximum lift.

In Part I the assumptions 1, 2 and 3 are retained, the theory being worked out for the case of infinitely heavy blades; this is equivalent to neglecting the "coning" angle, i.e., the constant term, in addition to the terms mentioned in 3 as being neglected in R. & M. 1111. It may be remarked that in practice the coning angle might be modified to any extent by offsetting the blade hinges, and could theoretically be reduced to zero whatever the weight of the blades. The direct method is first of all pursued of working out the force components (thrust and longitudinal force) as in R. & M. 1111. Afterwards it is found possible to determine the ratio of drag to lift from a consideration of the energy account; this method is found to be simpler than that of calculating the longitudinal force and gives identical results.

Part II.—The restriction to infinitely heavy blades is removed and the flapping motion is worked out in a general manner subject to assumptions 1 and 2. The general solution is obtained in the form of an infinite series of linear simultaneous equations for the coefficients. These equations might be solved by successive approximation without restriction, but in practice the solution has been obtained only (a) as an expansion in powers of μ and (b) as an expansion in power of a variable γ representing the ratio of the aerodynamic force to the weight of the blades. ($\gamma = c \rho a R^4 / I_1$, where a is the slope of the lift curve and I_1 is the moment of inertia of one blade about its hinge.) Since μ is generally less than 0.5, while γ has a value between 6.0 and 10.0 for the full scale machine it appears that the expansion in powers of μ is of the greater practical importance. It is proposed to work out a few standard numerical cases as far as terms in μ^4 in the equation of zero torque which includes terms in $\cos 2\psi$ and $\sin 2\psi$.

2. Introduction to Part I.—The present investigation was originally undertaken on account of a suggestion made independently by Mr. McKinnon Wood and Mr. Townend with regard to a possible mechanical alternative to the autogyro in which the blades, instead of being free to flap about hinges normal to the axle, are rotated about the blade axis by a suitable cam mechanism so as to change the blade angle periodically round the circle. It was argued that with a suitable design of cam the two machines are identical except

that they will fly with their axles inclined at different angles to the horizontal.* On the other hand, the formulae of R. & M. 1111, if applied directly and independently to the two machines, gave the result that the maximum lift/drag of the machine with variable blade angles is roughly double that of the machine with flapping blades, for the standard case of mean blade angle $\theta_0 = 2^\circ$, while the discrepancy increases with increasing blade angle. The results of the present report show that this discrepancy is entirely fictitious.

The physical argument is as follows :—

For an autogyro, with infinitely heavy straight blades free to flap, neglecting gravity (i.e., if the centrifugal is large compared with the aerodynamic force), each blade will move in a plane through the centre, but this plane will be inclined to the plane normal to the axis in such a way as to equalise the thrust moments of opposite blades about the centre. The inclination will be through an angle β_1 in the plane of symmetry, the sideways inclination being zero for heavy straight blades. If the plane of motion of the blades is chosen as axis of reference they may be considered as not flapping but as having a blade angle which varies periodically round the circle. The machine is therefore mechanically equivalent to a machine with non-flapping blades whose blade angles are varied in a particular way by means of a cam and whose axle is inclined at an angle β_1 to the axle of the original machine. The condition of zero torque must also be identical on the two machines as the mean torque is zero about any axis through the centre.

2.1. *Orders of Magnitude of the various terms.*—On the assumption that the discrepancy just mentioned is connected with the nature of the approximations made in R. & M. 1111, the first step is to define the order of magnitude of the various terms on some definite basis. In order to arrive at preliminary ideas as to which terms may be considered small, we start with the equations of R. & M. 1111, as they stand for the special case of zero blade angle θ .

Equation 29 of R. & M. 1111 becomes, for θ zero,

$$\delta = 2a x^2,$$

where $a (= 3)$ is the slope of the lift curve.

We shall consider $\sqrt{\frac{\delta}{2a}}$ ($= x_0$) as the standard small quantity of the first order. Its magnitude in the standard case of $\delta = 0.006$ is $x_0 = 0.032$.

Equation (23) then gives—

$$\beta_1 = 2 \mu x_0.$$

* This is not exactly true unless the blades are considered to be infinitely heavy and gravity is neglected.

where μ stands for $V \cos i/R \Omega$. Hence, β_1 is of order not greater than x_0 . Equation (25) becomes, for θ zero—

$$T = b c \rho \Omega^2 R^3 \cdot \frac{1}{2} a x_0,$$

while equation (31) becomes—

$$\begin{aligned} H &= b c \rho \Omega^2 R^3 \left\{ \frac{1}{2} \delta + \frac{3}{2} a x_0^2 \right\} \mu \\ &= b c \rho \Omega^2 R^3 \cdot \frac{5}{2} a x_0^2 \mu. \end{aligned}$$

The equation for interference flow (13) may be written—

$$\tan i = \frac{x_0}{\mu} + \frac{\frac{1}{4} \sigma a x_0}{\mu \sqrt{\mu^2 + x_0^2}} \dots \dots \dots (1)$$

and the equation for X/Z may be written—

$$\begin{aligned} \frac{X}{Z} &= \tan i + \frac{H}{T} \\ &= \frac{x_0}{\mu} + \frac{\frac{1}{4} \sigma a x_0}{\mu \sqrt{\mu^2 + x_0^2}} + 5 \mu x_0 \dots \dots (2) \end{aligned}$$

For this quantity to be a minimum for variations of μ it is necessary that at least two of the terms should be of the same order of magnitude, consequently μ must be considered as finite* in the neighbourhood of minimum X/Z , while $\tan i$ is a small quantity of order x_0 . On the other hand, when $\tan i$ is finite, equation (1) shows that μ is of the order $(x_0 \cot i)^{\frac{1}{2}}$.

But even when μ is small, equations 1 and 2 show that since terms of order μ^2 have been neglected in obtaining the equation $\delta = 2a x_0^2$ (the condition of zero torque), which determines x_0 as a function of δ (and μ), there is a possibility of an error of order μ in the value of $\tan i$ for given μ , and this is of the same order as the contribution to X/Z of the whole of the longitudinal force. Again, if k_z is the lift coefficient, equation 2 may be written—

$$\frac{X}{Z} = \frac{x_0}{\mu} + \frac{1}{2} k_z + 5 \mu x_0$$

to a sufficient approximation, and μ may be considered as a function of k_z given by the equation—

$$k_z = \frac{\frac{1}{2} \sigma a x_0}{\mu \sqrt{\mu^2 + x_0^2}}.$$

* Throughout the report any quantity is described as:—

small if it is of order x_0^p
finite if it is of order unity
large if it is of order x_0^{-p}

where p is positive.

In other words, in evaluating the drag X for a given lift Z or thrust T from the relation—

$$X = T \sin i + H \cos i$$

the approximation of R. & M. 1111 allows of errors in the evaluation of i for given T or Z , which are of the same order of importance as the whole contribution of H to the drag X . This is the real cause of the fictitious discrepancy between the results for the two machines.

All the above conclusions as to orders of magnitude still hold when the mean blade angle θ_0^* is not zero, provided that θ_0 is assumed to be a small quantity of order x_0 . For the standard autogyro $\theta_0 = 2^\circ = 0.035$ radians, so that the assumption is justified.

In the following analysis, therefore, x and μ will be assumed provisionally to be of order $x_0 \left(= \sqrt{\frac{\delta}{2a}} \right)$ and all terms of higher order than the principal terms will be neglected throughout as in R. & M. 1111. This assumption will be justified in the course of the analysis. On the other hand μ will be treated as finite† throughout. The early portions of the analysis are identical with R. & M. 1111, but are repeated here in order to verify the order of magnitude of the various terms neglected or retained.

3. *Velocity components.*—The fundamental assumption is that the resultant air velocity affecting the blade elements is constant in magnitude and direction over the airscrew disc, and may be resolved into components, $x R \Omega$ parallel to the axle, and $\mu R \Omega$ normal to the axle, where Ω is the angular velocity.

The spherical polar co-ordinates of the line of a blade at any instant are ψ and β (see Fig. 1) where $d\psi/dt = \Omega$. The component velocities of the air relative to a blade element at radius r are:—
 $x R \Omega - r \dot{\beta}$ parallel to the axle, $\mu R \Omega \cos \psi$ normal to the axle in a plane through the blade (i.e., along the radius) and $r \Omega + \mu R \Omega \sin \psi$ normal to this plane, neglecting squares of β . To the same approximation the air velocity relative to a blade element may be resolved into—

(1) Along the blade :— $\mu R \Omega \cos \Psi$

(2) Normal to the blade in a plane through the axle :—

$$U_y = x R \Omega - r \dot{\beta} - \mu R \Omega \beta \cos \Psi$$

* θ_0 is written for the mean blade angle to distinguish it from the actual blade angle θ which may vary round the circle.

† See footnote on page 599.

(3) Normal to this plane :—

$$U_x = r \Omega + \mu R \Omega \sin \Psi.$$

For a straight blade these are identical with the equations at the top of page 564, R. & M. 1111.

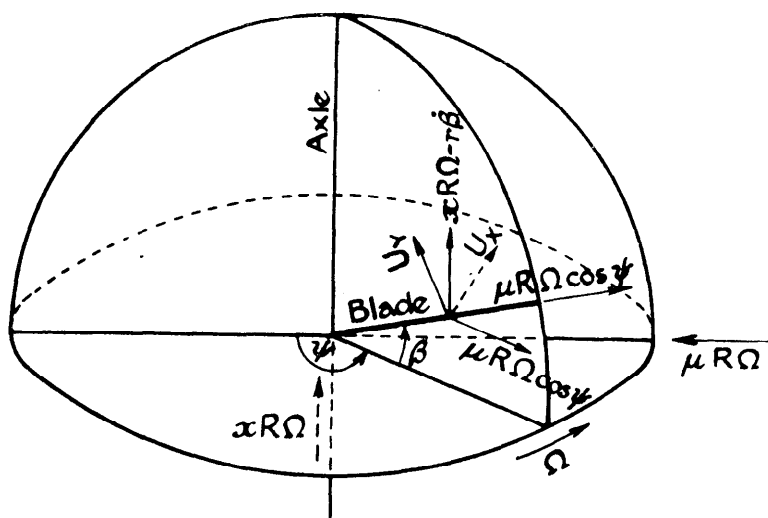


FIG. 1.

If U is the resultant component velocity in a plane normal to the blade, and ϕ the angle which it makes with the line mutually perpendicular to the blade and the axle, it follows that—

$$U \cos \phi = r \Omega + \mu R \Omega \sin \Psi = U_x, \quad \dots (3)$$

$$U \sin \phi = x R \Omega - r \dot{\beta} - \mu R \Omega \beta \cos \Psi = U_y. \quad (4)$$

(see Fig. 1).

The two terms in U_x are both finite and it follows, from the assumption that x and β are of order x_0 , that the three terms in U_y are of order x_0 , and that ϕ is of order x_0 over that part of the airscrew disc for which U_x is finite and positive. When U_x is finite and negative, $\pi - \phi$ is of order x_0 , while ϕ varies between 0 and π in the small area of the disc in which U_x is of the order x_0 .

4. *Aerodynamic forces.*—The assumptions are identical with those of R. & M. 1111.

(1) The resultant aerodynamic force on an element lies in a plane normal to the blade, and is a function of the component velocity in that plane only.

(2) The magnitude of the resultant force as expressed by lift and drag coefficients is given by—

$$k_L = a \alpha \text{ (for a symmetrical section),}$$

$$k_D = \delta,$$

where a and δ are constants,

$$\alpha = \phi + \theta,$$

where θ is the blade angle referred to a line mutually perpendicular to the blade and the axle. This assumption not only ignores the effect of the stall, but gives a drag of the wrong sign over the area of the disc for which U_x is negative.

Neglecting squares of x and β , the resultant force on an element of one blade may be resolved into—

(1) parallel to the axle :—

$$\left. \begin{aligned} d T_1 &= c \rho dr a \alpha U^2 \\ &= c \rho dr a (U_x U_y + \theta U_x^2) \end{aligned} \right\} \dots \dots (5)$$

(2) normal to the blade and the axle :—

$$\left. \begin{aligned} \frac{1}{r} d Q_1 &= c \rho dr (\delta U^2 - a \alpha \phi U^2) \\ &= c \rho dr \{ \delta U_x^2 - a \theta U_x U_y - a U_y^2 \} \end{aligned} \right\} (6)$$

(3) Radial outwards normal to the axle :—

$$- \beta d T_1. \dots \dots \dots (7)$$

The force components (2) and (3) resolved parallel and perpendicular to the direction of the velocity $\mu R \Omega$ in the plane of rotation, give—

$$\text{element of longitudinal force } d H_1 = \frac{1}{r} d Q_1 \sin \Psi - \beta d T_1 \cos \Psi,$$

$$\text{element of sideways force } d Y. = \frac{1}{r} d Q_1 \cos \Psi + \beta d T_1 \sin \Psi. \dots (8)$$

For the case of "heavy" straight blades, neglecting gravity, it will be assumed that the blades move in a plane through the centre inclined at an angle β_1 to the plane normal to the axle, so that :—

$$\left. \begin{aligned} \beta &= \beta_1 \cos (\Psi - \Psi_1) \\ \dot{\beta} &= \Omega \beta_1 \sin (\Psi - \Psi_1). \end{aligned} \right\} \dots \dots (9)$$

This will be verified in the next two sections where it will be shown that $\Psi_1 = 0$ (for "heavy" straight blades). Hence equations (3) and (4) for the velocity components may be written :—

$$U = U_x = r \Omega + \mu R \Omega \sin \Psi \dots \dots (10)$$

$$\phi U = U_y = x R \Omega - r \Omega \beta_1 \sin \Psi + \mu R \Omega \beta_1 \cos^2 \Psi^* (11)$$

* This term is neglected in R. & M. 1111.

The mean value round the circle of any force component F is given by—

$$dF = \frac{b}{2\pi} \int_0^{2\pi} dF_1 d\Psi,$$

(where b is the number of blades) and the mean value for the whole autogyro is then obtained by integrating along the blade from O to R .

4.1. *Equation of flapping motion.*—The equation for the steady flapping motion is obtained as in R. & M. 1111 (page 562) by taking moments about the hinge in the form—

$$I_1 (\ddot{\beta} + \Omega^2 \beta) = (TM)_1,$$

since squares of β are neglected, the blades are straight and gravity is neglected. $(TM)_1$ is the moment of the thrust on one blade about the hinge considered as a function of Ψ .

Expanding β and $(TM)_1$ as a Fourier's series in the form—

$$\beta = \beta_0 - \beta_1 \cos(\Psi - \Psi_1) - \beta_2 \cos 2(\Psi - \Psi_2) + \dots$$

$$(TM)_1 = \tau_0 - \tau_1 \cos(\Psi - \omega_1) - \tau_2 \cos 2(\Psi - \omega_2) + \dots$$

the equation becomes—

$$I_1 \Omega^2 \{ \beta_0 + 3 \beta_2 \cos 2(\Psi - \Psi_2) + 8 \beta_3 \cos 3(\Psi - \Psi_3) \} + \dots \\ = \tau_0 - \tau_1 \cos(\Psi - \omega_1) + \dots \quad \dots \quad \dots \quad (12)$$

It follows that if $I_1 \Omega^2$ is large compared with $(TM)_1$ we have to the first order—

$$\beta_0 = \beta_2 = \beta_3 \quad \dots \quad \dots = 0 \\ \text{and } \tau_1 \cos(\Psi - \omega_1) = 0. \quad \dots \quad \dots \quad (13)$$

The expression $\tau_1 \cos(\Psi - \omega_1)$, when evaluated by means of equations (5), (10) and (11), will be found to contain β_1 and Ψ_1 whose value may therefore be determined from the last equation.

On substituting the values of β_1 and Ψ_1 in τ_0 , τ_2 , etc., equation (12) would determine first approximations to β_0 , β_2 , etc., in the form—

$$\beta_0 = \frac{\tau_0}{I_1 \Omega^2}$$

$$\beta_2 = \frac{\tau_2}{3I_1 \Omega^2}, \quad \Psi_2 = \omega_2 \text{ etc.}$$

and the modified expression for β could then be substituted in the force equations.

In the following discussion of the force components equations are obtained for the two cases :—(A) a machine with flapping blades and constant blade angle ; (B) a machine in which the blade angle θ is caused to vary with Ψ in such a way that the flapping motion

disappears, for infinitely heavy blades. These two cases will be treated at first as though they were absolutely distinct from one another but subsequently it will appear that they are identical.

4.2 *Thrust moment (A), case of constant blade angle $\theta = \theta_0$.*—Equation (5) gives—

$$\begin{aligned} (T M)_1 &= \int_0^R r dT_1 \\ &= c \rho a \int_0^R (U_x U_y + \theta_0 U_x^2) r dr. \end{aligned}$$

Substituting from equations (10) and (11),

$$\begin{aligned} (TM)_1 &= c \rho a \int_0^R \{ (r \Omega + \mu R \Omega \sin \Psi) (x R \Omega - r \Omega \beta_1 \sin \Psi \\ &\quad + \mu R \Omega \beta_1 \cos^2 \Psi) + \theta_0 (r \Omega + \mu R \Omega \sin \Psi)^2 \} r dr. \end{aligned}$$

Expanding as a Fourier's series in Ψ :—

$$\text{Coefficient of } \cos \Psi (= \tau_1 \cos \omega_1) = 0$$

Coefficient of $\sin \Psi (= \tau_1 \sin \omega_1)$ is derived from the following terms of the integrand :—

$$\begin{aligned} \mu x r R^2 \Omega^2 \sin \Psi - \beta_1 r^3 \Omega^2 \sin \Psi + \mu^2 \beta_1 r R^2 \Omega^2 \sin \Psi \cos^2 \Psi \\ + 2 \mu \theta_0 r^2 R \Omega^2 \sin \Psi \end{aligned}$$

and is therefore equal to :—

$$\begin{aligned} c \rho a \int_0^R \left\{ \mu x r R^2 \Omega^2 + 2 \mu \theta_0 r^2 R \Omega^2 - \beta_1 (r^3 \Omega^2 \right. \\ \left. - \frac{1}{4} \mu^2 r R^2 \Omega^2) \right\}^* dr \\ = c \rho a R^4 \Omega^2 \left\{ \frac{1}{2} \mu x + \frac{2}{3} \mu \theta_0 - \beta_1 \left(\frac{1}{4} - \frac{1}{8} \mu^2 \right) \right\}^*. \end{aligned}$$

Hence, equation (13) is satisfied provided this last expression vanishes, leading to the relation—

$$\beta_1 = \frac{2 \mu \left(x + \frac{4}{3} \theta_0 \right)}{1 - \frac{1}{2} \mu^2} \quad \dots \quad (14A)$$

$$\Psi_1 = 0.$$

This verifies that if θ_0 is of order x_0 , and x is of order x_0 , then β_1 , β and β are of order x_0 .

4.21. *Thrust moment, B.*—The blade angle is supposed to vary with Ψ in such a way that—

$$\theta = \theta_0 - \theta_1 \sin \Psi.$$

* Terms in μ^2 are neglected in R. & M. 1111.

It will be shown that the value of θ_1 may be chosen so as to balance the thrust moment with β_1 zero.

The extra terms in the coefficient of $\sin \Psi$ in the expansion of the thrust moment are—

$$\begin{aligned} & -c \rho a \int_0^R \theta_1 \left(r^3 \Omega^2 + \frac{3}{4} \mu^2 r R^2 \Omega^2 \right) dr \\ & = -\theta_1 c \rho a R^4 \Omega^2 \left(\frac{1}{4} + \frac{3}{8} \mu^2 \right). \end{aligned}$$

Hence, if we put $\beta_1 = 0$ and

$$\theta_1 = \frac{2 \mu \left(x + \frac{4}{3} \theta_0 \right)}{1 + \frac{3}{2} \mu^2} \quad \dots \quad \dots \quad \dots \quad \dots \quad (14B)$$

the coefficient of $\sin \Psi$ vanishes and equation (13) is satisfied.

4.3. *Equation for the torque (A).*—The mean torque for the whole autogyro can be obtained by integrating equation (6) in the form

$$Q = \frac{b}{2\pi} \int_0^{2\pi} d\Psi c \rho \int_0^R r dr \left\{ \delta U_x^2 - a \theta_0 U_x U_y - a U_y^2 \right\}.$$

Substituting for U_x , U_y , from equations (10) and (11), it appears that the only terms of the integrand which could contribute to the mean torque are—

$$\begin{aligned} & c \rho \delta \left[r^3 \Omega^2 + \mu^2 r R^2 \Omega^2 \sin^2 \Psi \right] \\ & - c \rho a \theta_0 \left[x r^2 R \Omega^2 - \mu r^2 R \Omega^2 \beta_1 \sin^2 \Psi \right. \\ & \quad \left. + \mu r^2 R \Omega^2 \beta_1 \cos^2 \Psi \right] \\ & - c \rho a \left[x^2 r R^2 \Omega^2 + r^3 \Omega^2 \beta_1^2 \sin^2 \Psi \right. \\ & \quad \left. + 2 \mu x r R^2 \Omega^2 \beta_1 \cos^2 \Psi + \mu^2 r R^2 \Omega^2 \beta_1^2 \cos^4 \Psi \right]. \end{aligned}$$

Hence—

$$\begin{aligned} Q &= b c \rho \int_0^R \left\{ \delta \left[r^3 \Omega^2 + \frac{1}{2} \mu^2 r R^2 \Omega^2 \right] - a \theta_0 x r^2 R \Omega^2 \right. \\ & \quad \left. - a \left[x^2 r R^2 \Omega^2 + \frac{1}{2} r^3 \Omega^2 \beta_1^2 + \mu x r R^2 \Omega^2 \beta_1 \right. \right. \\ & \quad \quad \left. \left. + \frac{3}{8} \mu^2 r R^2 \Omega^2 \beta_1^2 \right] \right\} dr \\ &= b c \rho R^4 \Omega^2 \left\{ \frac{1}{4} \delta (1 + \mu^2) - \frac{1}{3} a x \theta_0 - a \left[\frac{1}{2} x^2 + \frac{1}{8} \beta_1^2 \right. \right. \\ & \quad \left. \left. + \frac{1}{2} \mu x \beta_1 + \frac{3}{16} \mu^2 \beta_1^2 \right] \right\} \dots \quad \dots \quad (15A). \end{aligned}$$

The condition of zero torque becomes therefore (on substituting for β_1 from equation 14A)—

$$\frac{\delta}{2a} (1 + \mu^2) \equiv x_0^2 (1 + \mu^2)$$

$$= \left\{ x \frac{1 + \frac{3}{2} \mu^2}{1 - \frac{1}{2} \mu^2} + \frac{8}{3} \theta_0 \frac{\mu^2}{1 - \frac{1}{2} \mu^2} \right\} \left\{ x \frac{1 + \frac{1}{2} \mu^2}{1 - \frac{1}{2} \mu^2} + \frac{2}{3} \theta_0 \frac{1 + \frac{3}{2} \mu^2}{1 - \frac{1}{2} \mu^2} \right\} \dots \dots (16A)^*$$

This is a quadratic equation to determine x for given values of x_0 , θ_0 and μ . Its form shows that in general, if θ_0 is of the order x_0 then x is of the order x_0 , thus verifying the assumptions made above that x , β and $\dot{\beta}$ are all of the order x_0 ($= \sqrt{\frac{\delta}{2a}}$).

4.31. Equation for the torque (B).—The formulae become—

$$Q = \frac{b}{2\pi} \int_0^{2\pi} d\Psi c \rho \int_0^R r dr \left\{ \delta U_x^2 - a \theta_0 U_x U_y + a \theta_1 U_x U_y \sin \Psi - a U_y^2 \right\}$$

with β_1 zero in equation (11)—

$$= bc \rho R^4 \Omega^2 \left\{ \frac{1}{4} \delta (1 + \mu^2) - \frac{1}{2} ax \left(\frac{2}{3} \theta_0 - \frac{1}{2} \mu \theta_1 + x \right) \right\} \dots (15B)$$

Substituting for θ_1 from equation (14b), the condition of zero torque becomes—

$$\frac{\delta}{2a} (1 + \mu^2) \equiv x_0^2 (1 + \mu^2) = x \left\{ x \frac{1 + \frac{1}{2} \mu^2}{1 + \frac{3}{2} \mu^2} + \frac{2}{3} \theta_0 \frac{1 - \frac{1}{2} \mu^2}{1 + \frac{3}{2} \mu^2} \right\} \dots \dots (16B).$$

* In Appendix 2, p. 618, the equations of the present report are summarised and compared with the corresponding equations of R. & M. 1111.

4.4. *Thrust, longitudinal force and cross wind force, Case A.*—The mean values of the thrust T , the longitudinal force H and the cross wind force Y may be obtained in a similar manner from equations (5), (8), (10) and (11).

The resulting formulae are—

$$T = b c \rho \Omega^2 R^3 \cdot \frac{1}{2} a \left\{ x + \frac{2}{3} \theta_0 \left(1 + \frac{3}{2} \mu^2 \right) \right\}, \dots (17A)$$

$$H = b c \rho \Omega^2 R^3 \left\{ \frac{1}{2} \mu \delta - \frac{1}{2} \mu a x \theta_0 + a \beta_1 \left[\frac{3}{4} x + \frac{1}{3} \theta_0 + \frac{1}{4} \mu \beta_1 \right] \right\}, \dots (18A)$$

$$Y = 0.$$

4.41. *Case B.*—The resulting formulae are—

$$T = b c \rho \Omega^2 R^3 \cdot \frac{1}{2} a \left\{ x + \frac{2}{3} \theta_0 \left(1 + \frac{3}{2} \mu^2 \right) - \mu \theta_1 \right\} (17B)$$

$$H = b c \rho \Omega^2 R^3 \left\{ \frac{1}{2} \mu \delta - \frac{1}{2} \mu a x \theta_0 + \frac{1}{4} a \theta_1 x \right\}, \dots (18B)$$

$$Y = 0.$$

5. *Identity of the Two Cases A and B.*—It will now be verified that the formulae of type A and B are connected by relations which correspond to a simple change of axes and are therefore equivalent.

Consider the result of resolving the air velocities relative to the screw parallel and perpendicular to the plane in which the blades move instead of to the axle. Quantities referred to the old and new axes are distinguished by suffixes A and B respectively. The new axes make an angle β_1 with the old and hence, neglecting squares and products of β_1 and x , we have—

$$\left. \begin{aligned} \mu_B &= \mu_A \\ x_B &= x_A + \mu \beta_1 \end{aligned} \right\} \dots \dots \dots (19)$$

Relative to the new plane of reference the flapping angle is always zero and the blade angle varies round the circle according to the formula—

$$\theta = \theta_0 - \beta_1 \sin \Psi.$$

It may be verified that if in equation (11)—

$$\phi_A U (=U_y) = x_A R \Omega - r \Omega \beta_1 \sin \Psi + \mu R \Omega \beta_1 \cos^2 \Psi$$

we substitute—

$$\begin{aligned} x_A &= x_B - \mu \beta_1 \\ \phi_A &= \phi_B - \beta_1 \sin \Psi, \end{aligned}$$

the equation reduces to—

$$\phi_B U = x_B R \Omega$$

$$\begin{aligned} \text{and we have } \alpha &= \theta_0 + \phi_A \\ &= \phi_B + \theta_0 - \beta_1 \sin \Psi. \end{aligned}$$

Thus, the velocity components in systems A and B are consistent if we write—

$$\begin{aligned} \mu_A &= \mu_B (= \mu) \\ x_A &= x_B - \mu \beta_1 \\ \theta_1 &= \beta_1, \end{aligned}$$

and these are consistent with the equations—

$$\begin{aligned} \beta_1 &= \frac{2 \mu \left(x_A + \frac{4}{3} \theta_0 \right)}{1 - \frac{1}{2} \mu^2} \\ &= \theta_1 = \frac{2 \mu \left(x_B + \frac{4}{3} \theta_0 \right)}{1 + \frac{3}{2} \mu^2}. \end{aligned}$$

For the total force components we have—

$$\begin{aligned} T_B &= T_A \cos \beta_1 + H_A \sin \beta_1, \\ H_B &= H_A \cos \beta_1 - T_A \sin \beta_1, \end{aligned}$$

or

$$\begin{aligned} T_B &= T_A, \\ H_B &= H_A - T \beta_1, \end{aligned}$$

to our approximation.

The torque requires closer consideration as the torque axes² A and B are inclined at the angle β_1 .

The formula for case A may be written—

$$\begin{aligned} (dQ_A)_1 &= c \rho r dr \{ \delta U^2 - a \alpha \phi_A U^2 \} \\ &= c \rho r dr \{ \delta U^2 - a \alpha \phi_B U^2 + a \alpha U^2 \beta_1 \sin \Psi \} \\ &= (dQ_B)_1 + r dT_1 \beta_1 \sin \Psi. \end{aligned}$$

Now β_1 has been chosen to make the coefficient of $\sin \Psi$ in the expansion of $\int_0^R r dT_1$ zero, and consequently the term $r dT_1 \sin \Psi$ contributes *nothing* to Q . Hence—

$$Q_A = Q_B.$$

All these results may be verified algebraically from the formulae already obtained.

6. *Interference flow.*—We have so far obtained formulae which determine the force system corresponding to given values of μ and x ; if we add the condition that the torque must be zero, the number of independent variables is reduced from two to one, and the value of x for given μ can be obtained by solving the quadratic equation (16) (A or B). (Equation B is the simpler to use in practice.)

It remains to connect the results with the velocity of the autogyro relative to the air at a distance from it, as defined by the velocity V and the incidence i . The assumption of R. & M. 1111 is adopted, that the induced velocity v is parallel to the axle and is given by formula 11 of R. & M. 1111, Section (3):—

$$v = \frac{T}{2\pi R^2 \rho V'}, \quad \dots \dots \dots (20)$$

where V' is the resultant velocity at the autogyro relative to it. Hence

$$\mu R \Omega = V \cos i^*, \quad \dots \dots \dots (21)$$

$$V'^2 = (x^2 + \mu^2) R^2 \Omega^2$$

and

$$x R \Omega = V \sin i - v.$$

Eliminating v and V' ,

$$\tan i = \frac{x}{\mu} + \frac{\frac{1}{2}T}{\pi R^2 \rho R^2 \Omega^2 \mu \sqrt{\mu^2 + x^2}}$$

which is identical with equation (13) of R. & M. 1111, Section 3.

Writing t for $\frac{T}{bc\rho R^3 \Omega^2}$ the equation becomes—

$$\tan i = \frac{x}{\mu} + \frac{\frac{1}{2}\sigma t}{\mu\sqrt{\mu^2 + x^2}} \dots \dots \dots (22)$$

It is now necessary to compare the formulae for the two cases represented by the equations with suffixes A and B respectively. Logically, the interference flow must be assumed to be normal to the plane of reference in each case so that the directions of v_A and v_B are inclined at an angle β_1 , while—

$$i_A = i_B - \beta_1 \quad \dots \dots \dots (23)$$

and

$$V_A = V_B \quad \dots \dots \dots (24)$$

* Up to this point $\mu R \Omega$ has been defined as the component velocity in the plane of reference affecting the blade elements and has not been assumed to be equal to $V \cos i$.

These results should be consistent (neglecting squares and products of x and β_1) with the equations—

$$\mu R \Omega = V_A \cos i_A, \dots \dots \dots (21A)$$

$$\tan i_A = \frac{x_A}{\mu} + \frac{\frac{1}{2} \sigma t}{\mu \sqrt{\mu^2 + x_A^2}}, \dots \dots \dots (22A)$$

and the corresponding equations with suffix A replaced by suffix B.

Rationalising equation (22) :—

$$(\mu \tan i - x)^2 (\mu^2 + x^2) = \frac{1}{4} \sigma^2 t^2 = K x_0^2,$$

where K is a finite quantity, or—

$$\begin{aligned} \mu^4 \tan^2 i - 2x \mu^3 \tan i + x^2 \mu^2 (1 + \tan^2 i) - 2 \mu x^3 \tan i + x^4 \\ = \frac{1}{4} \sigma^2 t^2 = K x_0^2. \dots \dots (25) \end{aligned}$$

Hence, it follows from the first term that $\mu^2 \tan i$ is of order x_0 : from the third term, that $\mu \tan i$ is finite or small (when $\tan i$ is large): and the last two terms on the left-hand side are negligible.

We can write—

$$\beta_1 = C_1 \mu x_B,$$

where C_1 is a finite quantity and hence from equation (19)—

$$x_A = x_B (1 - \mu^2 C_1),$$

and from equation (23)—

$$\begin{aligned} \tan i_A &= \frac{\tan i_B - \beta_1}{1 + \beta_1 \tan i_B} \\ &= \tan i_B \{ 1 - C_1 (\mu \cdot \tan i_B) x_B \} - C_1 \mu x_B. \end{aligned}$$

Hence, equations (21A) and (23) give—

$$\begin{aligned} V_A \cos i_A &= V_B \cos i_B \\ &= V_B \cos i_A (1 - \beta_1 \tan i_A) \\ &= V_B \cos i_A \{ 1 - C_1 (\mu \cdot \tan i_A) x_B \} \\ &= V_B \cos i_A (1 - \text{term of order } x_B). \end{aligned}$$

Again, in transforming equation (25A)—

$$\begin{aligned} \mu \tan i_A - x_A &= \mu \tan i_B (1 - C_1 \mu \tan i_B \cdot x_B) - C_1 \mu^2 x_B \\ &\quad - x_B + C_1 \mu^2 x_B \\ &= \mu \tan i_B (1 - C_1 \mu \tan i_B x_B) - x_B, \end{aligned}$$

and

$$\mu^2 + x_A^2 = \mu^2 + x_B^2 (1 - C_1 \mu^2)^2,$$

and it may be verified that equation (25A) becomes—

$$\mu^4 \tan^2 i_B - 2x_B \mu^3 \tan i_B + x_B^2 \mu^2 (1 + \tan^2 i_B) = 1/4 \sigma^2 t^2 + \text{terms of order } x_0^3$$

and is therefore identical with equation (25B) to our order of approximation.

In the neighbourhood of maximum lift/drag the above discussion may be simplified. From equation (22) it follows that μ is not small but finite* and therefore $\tan i_A$ is of order x_0 and equation (22A) may be written—

$$\tan i_A = \frac{x_A}{\mu} + \frac{1}{2} \frac{\sigma t}{\mu^2} \dots \dots \dots (26A)$$

Hence

$$\begin{aligned} V_A \cos i_A &= V_B \cos i_B + \beta_1 V_B \sin i_B \\ &= V_B \cos i_B + \text{terms order } x_0^2 \end{aligned}$$

and equation (26A) may be written—

$$\tan i_B - \frac{\beta_1}{\cos^2 i_B} = \frac{x_B}{\mu} - \beta_1 + \frac{1}{2} \frac{\sigma t}{\mu^2}$$

or

$$\tan i_B = \frac{x_B}{\mu} + \frac{1}{2} \frac{\sigma t}{\mu^2} + \text{terms of order } x_0^2,$$

so that equations with suffixes A and B are consistent as before.

7. *Lift and Drag.*—The lift Z and drag X are given by the formulae—

$$\begin{aligned} Z &= T \cos i - H \sin i, \\ X &= T \sin i + H \cos i. \end{aligned}$$

Since H is of order μx_0^2 and T is of order x_0 , it follows that

$$Z = T \cos i (1 - H \tan i/T).$$

$$= T \cos i (1 - \text{terms of order } x_0)$$

and

$$\begin{aligned} \frac{H}{Z} &= \frac{\tan i + \frac{H}{T}}{1 - \frac{H}{T} \tan i} \\ &= \left(\tan i + \frac{H}{T} \right) (1 + \text{terms of order } x_0). \end{aligned}$$

* See footnote on page 599.

Substituting from formula (22)—

$$\begin{aligned} \frac{X}{Z} &= \tan i + \frac{H}{T} \\ &= \frac{x}{\mu} + \frac{H}{T} + \frac{\frac{1}{2} \sigma t}{\mu \sqrt{\mu^2 + x^2}} \dots \dots \dots (27) \end{aligned}$$

It may be verified by direct algebra that equations (27A) and (27B) give identical values of X/Z .

8. *Energy Account.*—The following argument was given originally in T.2144a, Section 3.* The rate at which work is done on the autogyro is VX . Since the torque is zero and the flapping motion of the blades requires no power, there are only two channels in which the autogyro wastes energy:—

- (1) In the work done by each element of thrust dT in producing the component velocity of downwash v ; since v is assumed to be constant over the disc the total loss is vT .
- (2) The work wasted by each element in profile drag. The value of this for an element is $c \rho U^3 k_D dr$ and the mean value of the total is therefore—

$$\frac{b}{2\pi} \int_0^{2\pi} d\Psi \int_0^R \rho c U^3 k_D dr.$$

Hence, we have the equations—

$$VX = \frac{b}{2\pi} \int_0^{2\pi} d\Psi \int_0^R \rho c U^3 k_D dr + vT, \quad \dots (28)$$

$$\frac{X}{Z} = \frac{\frac{b}{2\pi} \int_0^{2\pi} d\Psi \int_0^R \rho c U^3 k_D dr}{VT \cos i} + \frac{v}{V \cos i} \dots (29)$$

Substituting for U from equation (10)—

$$U = r \Omega + \mu R \Omega \sin \Psi,$$

the value of the profile drag integral on the assumptions of the present paper is—

$$\frac{b}{2\pi} \int_0^{2\pi} d\Psi \int_0^R \rho c U^3 \delta dr = bc \rho R^4 \Omega^3 \cdot \frac{1}{4} \delta (1 + 3 \mu^2),$$

* T.2144a. By C. N. H. Lock. Notes on T.2144. The lift and drag of an autogyro at small angles of incidence. By H. Glauert. Unpublished.

and equation (28) becomes—

$$\frac{X}{Z} = \frac{\delta(1 + 3\mu^2)^*}{4\mu t} + \frac{\frac{1}{2}\sigma t}{\mu\sqrt{\mu^2 + x^2}} \dots \dots \dots (30)$$

On comparing with equation (27) it appears that the last terms on the R.H.S. are identical and it may be verified algebraically that—

$$\frac{\delta(1 + 3\mu^2)}{4\mu t} = \frac{x_A}{\mu} + \frac{H_A}{T_A} = \frac{x_B}{\mu} + \frac{H_B}{T_B}$$

on using the values of the quantities on the R.H.S. already obtained.† Hence, this expression for the energy loss makes unnecessary the somewhat complicated formulae for the longitudinal force.

If the torque were not zero it is interesting to notice that it would be necessary to add to the right-hand side of equation (30) the term $Q \Omega/VT \cos i$ and this equation would become—

$$\frac{X}{Z} = \frac{\delta(1 + 3\mu^2)}{4\mu t} + \frac{q}{\mu t} + \frac{\frac{1}{2}\sigma t}{\mu\sqrt{\mu^2 + x^2}}, \dots \dots (31)$$

where q is written for $Q/bc \rho R^4 \Omega^2$.

This equation may also be identified algebraically with equations (15), (17), (18) and (27), A or B.

An example of the application of this equation would be the case of a helicopter flying horizontally under its own power for which the drag X would be zero in equation (31). On substituting values of t and q this equation becomes a quadratic for x_A or x_B , but of rather more restricted application than the condition of zero torque since it involves a term containing the solidity.

9. *Effect of Stalling.*—It is probable that the true value of X/Z is necessarily greater than the value calculated by formulae (29) and (30) based on a value of δ which is a suitable mean value of k_D below the stall. For not only does the true value of k_D increase very rapidly above the stall, but according to formula (10) $U (=U_x)$ is negative and so the integrand $U^3 k_D$ is negative over a part of the range of integration, whereas the true value of the integrand $U^3 k_D$ is always positive. On the other hand, for moderate values of μ the error of the approximation is probably fairly small, both because angles of incidence above the stall occur only over a moderate proportion of the disc and also (in view of the fact that the third power of U

* This expression is equivalent to equation (c) in Appendix 1 of R. & M. 1111, which also contains a further approximation to take account of the component velocity along the blades.

† This explains the discrepancy mentioned in the last paragraphs of R. & M. 1111, Appendix 1.

appears in the integrand) because the values of U corresponding to large angles of incidence are relatively small. The order of magnitude of the effect of "stalling" could be determined in any particular case by evaluating graphically the integral appearing in equation (29) on the basis of the performance data of the aerofoil section.

10. *Numerical results.*—Calculations have been made by the formulae of the present report for the standard case of $\theta_0 = 2^\circ$, $\delta = 0.006$, $\sigma = 0.2$ and also for $\theta_0 = 0^\circ$ and $\theta_0 = 4^\circ$ for the same values of δ and σ . The results can easily be extended to any value of σ , while those for a given value of δ and θ_0 can be applied at once to any other values for which the ratio of δ^\dagger to θ_0 is unaltered; e.g., the results for $\delta = 0.006$, $\theta_0 = 4^\circ$ can be applied to the case of $\delta = 0.0015$, $\theta_0 = 2^\circ$.

For the standard case ($\theta_0 = 2^\circ$) the Table at the end of Part I contains values of all the relevant quantities, calculated for a series of even values of μ^2 , compared with the corresponding quantities calculated by the formulae of R. & M. 1111 for the case of infinitely heavy straight blades. In Fig. 2 curves are plotted of X/Z against $\sigma t/\mu^2$, which is approximately equal to k_z , for the three blade angles $\theta_0 = 0^\circ$, $\theta_0 = 2^\circ$, $\theta_0 = 4^\circ$ together with the corresponding curves evaluated by the formulae of R. & M. 1111 for case A and case B. Case A corresponds to the numerical results given in R. & M. 1111, Fig. 8.*

In the neighbourhood of minimum X/Z equation (30) may be written to a good approximation

$$\frac{X}{Z} = \frac{\delta(1 + 3\mu^2)}{4\mu t} + \frac{\frac{1}{2}\sigma t}{\mu^2}$$

and it is obvious that for given values of μ , δ , and σ this expression has a definite minimum value if the value of t for given μ is supposed to be adjusted by varying the blade angle. Numerical results are given in the following table for $\delta = 0.006$, $\sigma = 0.2$.

$\mu =$	0.15	0.2	0.3	0.4	0.5	0.6
Minimum $X/Z =$	0.436	0.290	0.168	0.117	0.091	0.076

For given μ , the minimum value of X/Z varies as $\delta^\dagger \sigma^\dagger$.

The minimum value is attained for $\mu = 0.2$ when $\theta_0 = 2^\circ$ and for $\mu = 0.3$ for $\theta_0 = 4^\circ$ approximately, and the improvement obtained up to $\mu = 0.5$ by increasing θ_0 above 4° is small and is almost certainly swamped by the effect of stalling. On the other hand the value of X/Z for given k_z (Fig. 2) decreases indefinitely towards the limit $X/Z = \frac{1}{2}k_z$ as the blade angle is increased.

* Approximately $\frac{\sigma t}{\mu^2} = k_z \sec^3 i$, so that the difference between the two quantities increases very rapidly for large values of i , but in the range which is chiefly of interest here, the difference is negligible.

Conclusions.—The curves of Fig. 2 show that the discrepancy between the results of the present report and those of R. & M. 1111 for flapping blades increases rapidly with the blade angle. For the standard case of $\theta_0 = 2^\circ$ at $k_z = 0.07$ (corresponding to $\mu = 0.4$ which is roughly the case of the full scale autogyro at its maximum flying speed of 67 miles per hour) the present theory gives $Z/X = 7.5$ against the value 5.8 given by R. & M. 1111 while for larger values of k_z the discrepancy is less.

It is claimed that the formulae of the present report give a sufficiently accurate solution of the problem for an autogyro with heavy blades on the basis of assumptions 1 and 2 of the introduction, and according to the discussion in the last section the value of Z/X obtained should represent an upper limit to the true value on account of the effects of stalling. Consequently, it is quite likely that the true values of Z/X may be as near to the results of R. & M. 1111 as to those of the present report. It seems likely that the value $Z/X = 10.0$ represents an extreme upper limit for $\delta = 0.006$, $\sigma = 0.2$. This value of Z/X corresponds to $\theta_0 = 4^\circ$, $\mu = 0.54$, or $\theta = 6^\circ$, $\mu = 0.47$.

APPENDIX I.

NOTATION.

All symbols are used in the same sense as in R. & M. 1111 except those marked with a (*).

Dimensions of Blades—

b^*	number of blades.
θ	angle of pitch.
θ_o^*	mean angle of pitch.
R	extreme radius.
r	radius to blade element.
c	chord of blade element.
$\sigma = \frac{bc}{\pi R}$	(the solidity).

Motion of Blades—

Ω	angular velocity of autogyro about its shaft.
Ψ	angular position of blade.
β	angular displacement of blade about its hinge.
β_1	If the blades are infinitely heavy they move in a plane inclined at an angle β_1 to the plane normal to the axle.

General Motion—

Velocity of air at blade elements relative to centre of autogyro has components :—

	$x R \Omega$ parallel to axle.
	$\mu^* R \Omega$ normal to axle.
	angle of incidence.
V	forward speed.
v	axial induced velocity.
V'	resultant velocity of air at blade element relative to centre of autogyro.
U	component in a plane normal to the blade of the resultant velocity of air at blade element relative to blade element.
ϕ	inclination of U to the plane normal to the axle.

Forces.—On autogyro :—

T =	thrust.
H	longitudinal force.
X	drag.
Y	lateral force.
Z	lift.
Q	torque.

On blade elements :—

δ	is mean profile drag coefficient of the blade element.
a^*	is slope of lift curve of blade element.

Force Coefficients—

$$t^* = T/b c \rho R^3 \Omega^2$$

$$q^* = Q/b c \rho R^4 \Omega^2$$

$$h^* = H/b c \rho R^3 \Omega^2$$

Miscellaneous.—Suffix A refers to the normal autogyro with (infinitely heavy straight) flapping blades.

Suffix B refers to a machine in which the blade angle is varied round the axle in such a manner as to eliminate the flapping motion.

Alternatively suffix B refers to the normal autogyro with the motion referred to the normal to the plane in which the blades move instead of to the axle.

The quantities which are different when referred to the two systems of axes are: χ, i, v, ϕ, H .

APPENDIX 2.

Contains a summary of the formulae of the present report compared with the corresponding formulae of R. & M. 1111 (for infinitely heavy straight blades). In formulae marked A the motion is referred to the axle of the autogyro as axis; in B it is referred to the normal to the plane in which the blades move. The formulae of R. & M. 1111 may be obtained by omitting terms of order μ^2 in comparison with those retained.

Use B throughout.

	Present report.	R. & M. 1111.
Condition of zero torque:—		
A	$\left[x_A \frac{1 + \frac{3}{2} \mu^2}{1 - \frac{1}{2} \mu^2} + \frac{8}{3} \theta_0 \frac{\mu^2}{1 - \frac{1}{2} \mu^2} \right] \times$	
	$\left[x_A \frac{1 + \frac{1}{2} \mu^2}{1 - \frac{1}{2} \mu^2} + \frac{2}{3} \theta_0 \frac{1 + \frac{3}{2} \mu^2}{1 - \frac{1}{2} \mu^2} \right] \cdot$	
B	$x_B \left[\frac{1 + \frac{1}{2} \mu^2}{1 + \frac{2}{3} \mu^2} + \frac{2}{3} \theta_0 \cdot \frac{1 - \frac{1}{2} \mu^2}{1 + \frac{2}{3} \mu^2} \right]$	$\frac{\delta}{2a} = x_0^2 = x \left(x + \frac{2}{3} \theta_0 \right)$
Flapping angle β_1 A	$\frac{2 \mu \left(x_A + \frac{4}{3} \theta_0 \right)}{1 - \frac{1}{2} \mu^2}$	
B	$\frac{2 \mu \left(x_B + \frac{4}{3} \theta_0 \right)}{1 + \frac{3}{2} \mu^2}$	$\beta_1 = 2 \mu \left(x + \frac{4}{3} \theta_0 \right)$

$$\text{Thrust} \quad \dots \quad \dots \quad \text{.. A} \quad \left\{ \begin{array}{l} bc \rho \Omega^2 R^3 \cdot \frac{1}{2} a \left[x_A + \frac{2}{3} \theta_0 \left(1 + \frac{3}{2} \mu^2 \right) \right] \\ bc \rho \Omega^2 R^3 \cdot \frac{1}{2} a \left[x_B \right. \\ \left. + \frac{2}{3} \theta_0 \left(1 + \frac{3}{2} \mu^2 \right) - \mu \beta_1 \right] \end{array} \right\} \quad \text{T} = bc \rho \Omega^2 R^3 \frac{1}{2} a \left(x + \frac{2}{3} \theta_0 \right)$$

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$$\text{Longitudinal Force} \quad \dots \quad \dots \quad \text{.. A. } H_A = bc \rho \Omega^2 R^3 \left[\frac{1}{2} \mu \delta - \frac{1}{2} \mu a x_A \theta_0 \right. \\ \left. + a \beta_1 \left(\frac{3}{4} x_A + \frac{1}{3} \theta_0 + \frac{1}{4} \mu \beta_1 \right) \right] \\ H_A = bc \rho \Omega^2 R^3 \left[\frac{1}{2} \mu \delta \right. \\ \left. - \frac{1}{2} \mu a x \theta_0 \right. \\ \left. + a \beta_1 \left(\frac{3}{4} x + \frac{1}{3} \theta_0 \right) \right]$$

$$\text{.. B. } H_B = bc \rho \Omega^2 R^3 \left[\frac{1}{2} \mu \delta - \frac{1}{2} \mu a x_B \theta_0 + \frac{1}{4} a \beta_1 x_B \right] \\ H_B = bc \rho \Omega^2 R^3 \left[\frac{1}{2} \mu \delta \right. \\ \left. - \frac{1}{2} \mu a x \theta_0 + \frac{1}{4} a \beta_1 x \right]$$

APPENDIX 2—continued.

Present report.			R. & M. 1111.
Cross wind Force A & B.	$Y = 0$	$Y = 0$
Interference flow A.	$\tan i_A = \frac{x_A}{\mu} + \frac{\frac{1}{2} \sigma t}{\mu \sqrt{\mu^2 + x_A^2}}$	$\left\{ \begin{array}{l} \tan i_A = \tan i_B = \\ \frac{x}{\mu} + \frac{\frac{1}{2} \sigma t}{\mu \sqrt{\mu^2 + x^2}} \\ t = \frac{T}{bc \rho \Omega^2 R^3} \end{array} \right.$
 B.	$\tan i_B = \frac{x_B}{\mu} + \frac{\frac{1}{2} \sigma t}{\mu \sqrt{\mu^2 + x_B^2}}, \left(t = \frac{T}{bc \rho \Omega^2 R^3} \right)$	
Drag/Lift A.	$\frac{X}{Z} = \left\{ \begin{array}{l} \tan i_A + \frac{H_A}{T} \\ \tan i_B + \frac{H_B}{T} \end{array} \right\} = \frac{\delta(1 + 3\mu^2)}{4\mu t} + \frac{\frac{1}{2} \sigma t}{\mu \sqrt{\mu^2 + x^2}}$ (from energy account)	$\left(\frac{X}{Z} \right)_A = \tan i + \frac{H_A}{T}$
 B.		$\left(\frac{X}{Z} \right)_B = \tan i + \frac{H_B}{T}$

The above equations are sufficient to determine all the variables in terms of μ , x_0 and θ_0 , and if x_0 and θ_0 are considered as small quantities of the same order then all the above equations are consistent with one another and with the relations

$$\mu = V \cos i_A = V \cos i_B, \quad i_A = i_B - \beta_1, \quad x_A = x_B - \mu \beta_1,$$

$$H_A = H_B + T \beta_1.$$

TABLE

Results for the standard Autogyro for which $\delta = .006$, $\theta = 2^\circ$, $\sigma = 0.2$ (infinitely heavy straight blades).

μ	μ^2	x_A $10^{-2} \times$	x_B $10^{-2} \times$	β_1	$\frac{\sigma t}{T} =$ $\frac{\pi R^2 \rho R^2 \Omega^2}{\pi \rho R^3 \Omega V \cos i}$	$\frac{H_A}{\pi \rho R^3 \Omega V \cos i}$	$\frac{H_B}{\pi \rho R^3 \Omega V \cos i}$	$\frac{Z}{\pi R^2 \rho V^2}$	i_A	i_B	$\left(\frac{X}{Z}\right)_A$	$\left(\frac{X}{Z}\right)_B$
0.633	0.40	-0.53	3.61	3.76°	0.096	0.00177	0.00078	0.0239	0.2°	3.9°	0.121	
0.565	0.32	-0.13	3.34	3.51°	0.100	0.00186	0.00079	0.0312	0.8°	4.3°	0.119	
0.490	0.24	+0.30	3.07	3.19°	0.106	0.00198	0.00080	0.0437	1.6°	4.9°	0.122 ^s	
0.447	0.20	0.59	2.92	2.99°	0.109	0.00207	0.00080	0.0539	2.3°	5.3°	0.126	
0.400	0.16	0.86	2.78	2.75°	0.113	0.00216	0.00081	0.0696	3.2°	6.0°	0.133 ^s	
0.347	0.12	1.15	2.64	2.46°	0.117	0.00226	0.00081	0.0962	4.7°	7.1°	0.149	
0.283	0.08	1.48	2.50	2.07°	0.123	0.00238	0.00082	0.1435	7.3°	9.5°	0.184	
0.200	0.04	1.83	2.36	1.51°	0.129	0.00253	0.00082	0.2910	14.1°	15.6°	0.292	

Formulae of present report.

Formulae of R. & M. 1111.

0.633	0.40	2.20	2.20	4.95°	0.137	0.00270	0.00082	0.0339	2.9°	2.9°	0.177	0.090
0.565	0.32	2.20	2.20	4.45°	0.137	0.00270	0.00082	0.0423	3.4°	3.4°	0.171 _s	0.094 _s
0.490	0.24	2.20	2.20	3.85°	0.137	0.00270	0.00082	0.0564	4.2°	4.2°	0.170	0.103
0.447	0.20	2.20	2.20	3.52°	0.137	0.00270	0.00082	0.0675	4.8°	4.8°	0.172	0.110
0.400	0.16	2.20	2.20	3.14°	0.137	0.00270	0.00082	0.0842	5.6°	5.6°	0.176 _s	0.122
0.347	0.12	2.20	2.20	2.72°	0.137	0.00270	0.00082	0.1116	6.9°	6.9°	0.188 _s	0.141
0.283	0.08	2.20	2.20	2.22°	0.137	0.00270	0.00082	0.1648	9.3°	9.3°	0.219	0.180
0.200	0.04	2.20	2.20	1.57°	0.137	0.00270	0.00082	0.3075	15.7°	15.7°	0.320	0.293

PART II.

A GENERAL TREATMENT OF THE FLAPPING MOTION.

May, 1927.

Summary.—Introductory.—In Part I the theory of the autogyro given by Mr. Glauert in R. & M. 1111 was extended by taking account of squares and higher powers of the variable μ (the ratio of the component velocity in the plane of rotation to the tip speed). This analysis applied to the case of infinitely heavy straight blades, neglecting gravity. In Part II the case of curved blades of finite weight is treated in a general manner.

Range of Investigation.—The coefficients in the Fourier's expansion of the flapping angle appear as the variables in a set of linear simultaneous equations which can be solved in succession. The effect of the flapping on the lift and drag is obtained by substituting the coefficients (which are linear functions of x and θ_0) in the equation of zero torque, which has the form of a quadratic equation in x in the general case. The effect of the flapping angle on the equation for mean thrust is confined to a single term of the second order, while the equation of energy loss is unaltered. The equations are solved in detail by expanding in powers of μ as far as μ^4 in the equation of zero torque for straight blades and as far as terms of order μ^2 in the curvature. This includes the effect of the coefficients of $\cos 2\Psi$ and $\sin 2\Psi$ in the expansion of the flapping angle.

Conclusions.—The effect of the coning angle and the curvature of a blade is worked out for the full scale machine and is found to be by no means negligible. For blades having the form of a circular arc, it is found that if the coning angle is measured from the tangent line to the blade at the root, the additional effect of curvature is negligible. It follows that the effect of coning and curvature could be reduced to a small quantity by giving the blade a suitable curvature in the opposite sense to that of existing machines. The effect of a curvature of the airstream of the kind postulated in R. & M. 1111, Section 10, is similar to the effect of a curvature of the blades in the form of a circular arc referred to the tangent at the root as zero line, and has by the same argument a negligible effect on the lift and drag in agreement with the conclusion of R. & M. 1111. The second order terms (coefficients of $\cos 2\Psi$ and $\sin 2\Psi$ in the flapping angle) produce an effect on the lift and drag which is negligible throughout the working range of the full scale machine and is a fortiori negligible for models, since the blades are heavier.

1. *Introduction.*—The object of the following analysis is to remove the restriction to infinitely heavy blades (which was imposed on the results of Part I) by treating the flapping motion in a general manner. This is the more necessary as it appears on examination that the coefficient defining the ratio of the inertia to the aerodynamic forces is by no means a small quantity in practice. The problem of treating the flapping motion in a general manner on the basis of the assumptions (1) and (2) of Part I is purely a matter of somewhat complicated algebra, but before proceeding with the analysis the following considerations are of interest.

On page 612 of Part I, in obtaining an expression for the energy loss (equation 30), it is stated that the flapping motion of the blades requires no power; this must be true in general and not only in

the case where the blades are infinitely heavy. In the equation of energy loss equation (30) Part I, in the neighbourhood of maximum lift/drag, x^2 is negligible in comparison with μ^2 and the equation may be written—

$$\frac{X}{Z} = \frac{\delta(1 + 3\mu^2)}{4\mu t} + \frac{1}{2} \frac{\sigma t}{\mu^2} \dots \dots \dots (1)$$

Using the approximate relation lift coefficient $k_z = \sigma t / \mu^2$, it follows that for a given lift coefficient (since δ and σ are constants) this expression for X/Z can only be affected by the flapping motion through the thrust coefficient t . Turning to equation (17A), which may be written—

$$t = \frac{1}{2} a \left\{ x + \frac{2}{3} \theta_0 \left(1 + \frac{3}{2} \mu^2 \right) \right\}, \dots \dots (2)$$

it appears on examination that this equation has the same form (for infinitely heavy blades) whether the blades are fixed, or are free to flap. It follows that the whole effect of flapping (for heavy blades) is due to the effect on the variable x of the additional terms in the equation of zero torque (15a). The terms involving β_1 in equation (15a) all involve positive coefficients of x and x^2 , and in general it follows that x and t are smaller for the case of blades free to flap than for fixed blades, and so from equation (1) the value of X/Z for given k_z is in general greater for blades free to flap.

Turning to the general case of flapping, it will be found that the general equation for t (equation 14 below) contains only one additional term (involving the coefficient of $\sin 2\Psi$ in β). Apart from this term the entire effect of flapping is still given by the equation of zero torque. It will be shown further that all coefficients in the Fourier's expansion of β are linear functions of x and θ_0 (homogeneous if the blades are straight) of which the coefficients are functions of the variables μ and γ only (for straight blades) where $\gamma = \frac{c \rho a R^4}{I_1}$, I_1 being the moment of inertia of one blade about its hinge.

The corresponding terms in the equation of zero torque are all quadratic functions of x and θ_0 (homogeneous for straight blades). Thus, all the terms due to flapping are of the same order in x_0 as the terms which are independent of flapping (see Part I, page 599). The value of x , and the complete solution for given μ can therefore be obtained by solving this quadratic equation.

The coefficients of the Fourier's expansion of β are obtained by expanding all terms in the general equation for the thrust moment as Fourier's series and equating coefficients. The result is a set of simultaneous equations which are linear in the coefficients and which could be solved by successive approximation. The values of the coefficients could then be substituted in the general equation

for zero torque. Actually the labour of computation increases very rapidly with increase in the order of the terms included, and it has been found necessary to expand the coefficients in powers of μ . A few numerical cases have been worked out as far as terms in μ^4 in the equation of zero torque; this includes coefficients as far as those of $\cos 2\Psi$, $\sin 2\Psi$.

2. *General analysis of the flapping motion.*—The whole of the present analysis is referred to the axle of the autogyro as axis of reference so that the axes correspond to axes A of Part I.

The method of obtaining the velocity components at a blade element and the components of force on the element, is identical with that of Part I. The equations for the velocity components are identical with equations 1, 3 and 4, except for the addition of a term for the curvature of the blade in equation 4, represented by the slope χ of the tangent to the blade at radial distance r referred to some definite zero line. Equation 4 is then identical in form with equation (2) on page 7 of R. & M. 1111 :—

$$U_x = r \Omega + \mu R \Omega \sin \Psi, \quad \dots \dots \dots (3)$$

$$U_y = x R \Omega - r \dot{\beta} - \mu R \Omega (\beta + \chi) \cos \Psi, \quad (4)$$

The elements of thrust and torque on one blade are :—

$$dT_1 = c \rho a dr (U_x U_y + \theta_0 U_x^2), \quad \dots \dots \dots (5)$$

$$\frac{1}{r} dQ_1 = c \rho dr \left\{ \delta U_x^2 - a \theta_0 U_x U_y - a U_y^2 \right\}, \quad \dots (6)$$

which are identical with the equations 5 and 6 of Part I.

Write—

$$\beta = a_0 - a_1 \cos \Psi - b_1 \sin \Psi \\ - a_n \cos n \Psi - b_n \sin n \Psi^*; \quad \dots \dots \dots (7)$$

then—

$$\dot{\beta} = a_1 \Omega \sin \Psi - b_1 \Omega \cos \Psi + 2a_2 \Omega \sin 2\Psi + \dots \\ + n a_n \Omega \sin n \Psi - n b_n \Omega \cos n \Psi. \quad \dots \dots (8)$$

Of the two velocity components, U_x is independent of β ; write—

$$U_y = x R \Omega + e_0 + e_1 \cos \Psi + f_1 \sin \Psi + \dots + e_n \cos n \Psi \\ + f_n \sin n \Psi, \quad \dots \dots (9)$$

* The notation of R. & M. 1111 and of Part I has been altered as it was not convenient for the general case.

The equation representing the condition of zero thrust moment about the hinge may be obtained as in Part I; it is identical with the equation of R. & M. 1111 (page 562, line 4), and will be written in the form:—

$$\frac{1}{\Omega^2} \cdot \ddot{\beta} + \beta = \frac{\gamma}{c \rho a R^4 \Omega^2} \int_0^R r dT_1 - C, \quad \dots \quad (12)$$

where:— $\gamma = c \rho a R^4 / I_1$, $C = G_1 / I_1 \Omega^2 + J_1 / I_1$.

I_1 is the moment of inertia of one blade about its hinge, J_1 is the product of inertia of the blade referred to the zero line of β , and G_1 is the moment of gravity about the hinge. (See R. & M. 1111, page 561, *et seq.*).

Substitute for $e_0 e_1$, etc., in terms of $a_0 a_1$, etc., in equation (11) and reduce to a Fourier's expansion; integrate with respect to r to determine $\int_0^R r dT_1$; substitute for β and $\ddot{\beta}$ from equation (7);

then both sides of (12) are expressed as Fourier's expansions in Ψ . Equating coefficients the following set of equations is obtained:—

From constant term:—

$$a_0 + C = \gamma \left\{ \frac{1}{3} x + \frac{1}{4} \theta_0 (1 + \mu^2) + \frac{1}{8} \mu b_2^* \right\}$$

from coefficient of $\cos \Psi$:—

$$0 = -\mu \left(\frac{1}{3} a_0 - 2 \eta_2^\dagger \right) + \frac{1}{4} b_1 \left(1 + \frac{1}{2} \mu^2 \right) - \frac{1}{6} \mu a_2 + \frac{1}{8} \mu^2 b_3^* \quad (13)$$

from coefficient of $\sin \Psi$:—

$$0 = \frac{1}{2} \mu x + \frac{2}{3} \mu \theta_0 - \frac{1}{4} a_1 \left(1 - \frac{1}{2} \mu^2 \right) - \frac{1}{6} \mu b_2 - \frac{1}{8} \mu^2 a_3^*$$

* It will be shown later that the terms underlined.....are of a higher order in μ than the remaining terms.

† Definitions—

$$-2 R^3 \eta_2 = \int_0^R r^2 \chi dr$$

$$-R^2 \eta_1 = \int_0^R r \chi dr$$

as in R. & M. 1111.

from coefficient of $\cos 2 \Psi$:—

$$(2^2 - 1) a_2 = \gamma \left\{ -\frac{1}{4} \mu^2 \theta_0 + \frac{2}{6} \mu a_1 + \frac{2}{4} b_2 \right. \\ \left. - \frac{2}{6} \mu a_3 + \frac{1}{8} \mu^2 b_4^* \right\}$$

from coefficient of $\sin 2 \Psi$:—

$$(2^2 - 1) b_2 = \gamma \left\{ -\mu^2 \left(\frac{1}{4} a_0 - \frac{1}{2} \eta_1 \right) + \frac{2}{6} \mu b_1 \right. \\ \left. - \frac{2}{4} a_2 - \frac{2}{6} \mu b_3 - \frac{1}{8} \mu^2 a_4 \right\}$$

from coefficient of $\cos n \Psi$:—

$$(n^2 - 1) a_n = \gamma \left\{ -\frac{1}{8} \mu^2 b_{n-2} + \frac{n}{6} \mu a_{n-1} + \frac{n}{4} b_n \right. \\ \left. - \frac{n}{6} \mu a_{n+1} + \frac{1}{8} \mu^2 b_{n+2} \right\} \dagger$$

from coefficient of $\sin n \Psi$:—

$$(n^2 - 1) b_n = \gamma \left\{ \frac{1}{8} \mu^2 a_{n-2} + \frac{n}{6} \mu b_{n-1} - \frac{n}{4} a_n \right. \\ \left. - \frac{n}{6} \mu b_{n+1} - \frac{1}{8} \mu^2 a_{n+2} \right\}$$

(13)
contd.

The only practicable method of solving this system of equations is to expand either in powers of γ or in powers of μ ; the latter method is preferred since γ is always much larger than μ in practice. It can be easily verified by induction that a_n, b_n are of the order μ_n ; it follows that the terms underlined.....in any equation are of higher order in μ than the remainder. Neglecting these terms, the n th pair of equations determine a_n and b_n in terms of coefficients of lower order and the equations can be solved in succession. It is obvious that the coefficients a_n, b_n , are homogeneous linear functions of x, θ_0 and the terms C, η_1, η_2 derived from the curvatures.

* It will be shown later that the terms underlined.....are of a higher order in μ than the remaining terms.

† The equations from $\cos 3 \Psi$ and $\sin 3 \Psi$ are of the general form.

It is interesting to notice that if the coefficients are expanded in powers of γ the order of the various terms is according to the following scheme :—

a_1 is of order unity,

a_0, b_1, a_2, b_3 are of order γ ,

b_2, a_3, b_4, a_5 are of order γ^2 ,

a_4, b_5, a_6, b_7 are of order γ^3 ,

etc.

3. *Equations for the thrust and torque.*—To obtain the effect of the flapping on the performance of an autogyro it is only necessary to work out the expressions for the thrust and torque by substituting for U_x, U_y in equations (5) and (6).

Thrust.—The only terms which contribute to the mean thrust are :—

$$\frac{dT_1}{dr} = c \rho a \left\{ r \Omega (x R \Omega + e_0) + \mu R \Omega f_1 \sin^2 \Psi \right. \\ \left. + \theta_0 (r^2 \Omega^2 + \mu^2 R^2 \Omega^2 \sin^2 \Psi) \right\}$$

and give

$$T = bc \rho R^3 \Omega^2 t = bc \rho a R^3 \Omega^2 \left\{ \frac{1}{2} x + \frac{1}{4} \mu^2 b_2 \right. \\ \left. + \frac{1}{3} \theta_0 \left(1 + \frac{3}{2} \mu^2 \right) \right\}. \quad (14)$$

Torque.—The only terms which contribute to the mean torque are :—

$$\frac{dQ_1}{dr} = c \rho \left\{ \delta (r^3 \Omega^2 + \mu^2 r R^2 \Omega^2 \sin^2 \Psi) \right. \\ \left. - a \theta_0 \left[r^2 \Omega (x R \Omega + e_0) + \mu r R \Omega f_1 \sin^2 \Psi \right] \right. \\ \left. - a \left[r (x R \Omega + e_0)^2 + r e_1^2 \cos^2 \Psi + r f_1^2 \sin^2 \Psi \right. \right. \\ \left. \left. + r e_2^2 \cos^2 2 \Psi + r f_2^2 \sin^2 2 \Psi + \dots \dots \dots \right] \right\}$$

and give

$$Q = \frac{b}{2\pi} \int_0^{2\pi} d\Psi \int_0^R \frac{dQ_1}{dr} dr \\ = bc \rho R^4 \Omega^2 \left\{ \frac{1}{4} \delta (1 + \mu^2) - a \theta_0 \left[\frac{1}{3} x + \frac{1}{R^4 \Omega} \int_0^R r^2 e_0 dr \right. \right. \\ \left. \left. + \frac{1}{2} \frac{\mu}{R^3 \Omega} \int_0^R r f_1 dr \right] \right\}$$

(contd. on next page)

$$- a \left[\frac{1}{2} x^2 + \frac{2x}{R^3 \Omega} \int_0^R r e_0 dr + \int_0^R r \left(e_0^2 + \frac{1}{2} e_1^2 + \frac{1}{2} f_1^2 + \frac{1}{2} e_2^2 + \frac{1}{2} f_2^2 + \dots \right) dr \right] \}.$$

Hence the equation of zero torque becomes

$$\frac{\delta}{2a} (1 + \mu^2) = x \left(x + \frac{2}{3} \theta_0 \right) + \mu x a_1 + \frac{1}{4} \mu^2 \theta_0 b_2 + \frac{1}{R^4 \Omega^2} \int_0^R r (2 e_0^2 + e_1^2 + f_1^2 + e_2^2 + f_2^2 + \dots) dr \dots \quad (15)$$

4. *Solution of the equation of zero torque by expanding in powers of μ as far as μ^4 .*

Case of straight blades neglecting gravity.—On examining the equation of zero torque it appears that in order to include all terms of order μ^4 in that equation it is necessary to evaluate the coefficients to the following order :—

- a_0 as far as μ^2 ,
- a_1 and b_1 as far as μ^3 ,
- a_2 and b_2 as far as μ^2 ,

Hence the following simplified forms of equations (13) are sufficient for the present purpose

$$\left. \begin{aligned} a_0 &= \gamma \left\{ \frac{1}{3} x + \frac{1}{4} \theta_0 (1 + \mu^2) \right\}, \\ a_1 &= \bar{a}_1 - \frac{2}{3} \mu b_2, \\ \bar{a}_1 &= \frac{2 \mu \left(x + \frac{4}{3} \theta_0 \right)}{1 - \frac{1}{2} \mu^2}, \dots \dots \dots \end{aligned} \right\} \quad (16)$$

so that \bar{a}_1 is identical with β_1 of Part I,

$$b_1 = \frac{\frac{4}{3} \mu a_0}{1 + \frac{1}{2} \mu^2} + \frac{2}{3} \mu a_2,$$

and a_2, b_2 are given by the equations

$$\left. \begin{aligned} 3a_2 - \frac{1}{2} \gamma b_2 &= \gamma \left(-\frac{1}{4} \mu^2 \theta_0 + \frac{1}{3} \mu a_1 \right), \\ \frac{1}{2} \gamma a_2 + 3 b_2 &= \gamma \left(-\frac{1}{4} \mu^2 a_0 + \frac{1}{3} \mu b_1 \right). \end{aligned} \right\} \quad (17)$$

In the last two equations it is sufficiently accurate to substitute

$$a_0 = \gamma \left(\frac{1}{3} x + \frac{1}{4} \theta_0 \right),$$

$$a_1 = 2 \mu \left(x + \frac{4}{3} \theta_0 \right),$$

$$b_1 = \frac{4}{3} \mu a_0.$$

The result of performing the substitutions and solving for a_2 and b_2 is:—

$$\left. \begin{aligned} \left(9 + \frac{1}{4} \gamma^2 \right) a_2 &= \mu^2 \left\{ -\frac{3}{4} \gamma \theta_0 + \gamma a_1 / \mu + \frac{7}{72} \gamma^2 a_0 \right\} \\ &= \mu^2 \gamma \left\{ x \left(2 + \frac{7}{216} \gamma^2 \right) + \theta_0 \left(\frac{23}{12} + \frac{7}{288} \gamma^2 \right) \right\}, \\ \left(9 + \frac{1}{4} \gamma^2 \right) b_2 &= \mu^2 \left\{ \frac{1}{8} \gamma^2 \theta_0 - \frac{1}{6} \gamma^2 a_1 / \mu + \frac{7}{12} \gamma a_0 \right\} \\ &= -\mu^2 \gamma^2 \left\{ \frac{5}{36} x + \frac{25}{144} \theta_0 \right\}. \end{aligned} \right\} \quad (18)$$

It is now necessary to substitute these values of the coefficients in the equations of zero torque and it will be found possible and convenient to analyse the equation in the following form:—

$$\frac{\delta}{2a} (1 + \mu^2) = F_1(x) + \mu^2 F_2(x) + \mu^2 F_3(x) + \mu^4 F(x) + \mu^4 F_5(x), \quad \dots \quad (19)$$

where:

$F_1(x)$ contains x and θ_0 only,

$F_2(x)$ contains \bar{a}_1 as a factor and does not contain a_2 or b_2 ,

$F_3(x)$ contains a_0 as a factor and does not contain a_2 or b_2 ,

$F_4(x)$ contains b_2 as a factor,

$F_5(x)$ contains a_2 as a factor.

It will be found that

$$F_1(x) = x \left(x + \frac{2}{3} \theta_0 \right),$$

$$F_2(x) + F_4(x) = \mu x a_1 + \frac{1}{4} \mu^2 \theta_0 b_2 + \frac{1}{R^4 \Omega^2} \int_0^R r (2e_0^2 + f_1^2 + e_2^2) dr,$$

so that

$$\left. \begin{aligned} F_2(x) &= \bar{a}_1 \left\{ \mu x + \frac{1}{4} \bar{a}_1 \left(1 + \frac{3}{2} \mu^2 \right) \right\}, \\ F_4(x) &= b_2 \left\{ -\frac{2}{3} \mu^2 x + \frac{1}{4} \mu^2 \theta_0 + b_2 \right\}; \end{aligned} \right\} \quad ($$

similarly

$$F_3(x) + F_5(x) = \frac{1}{R^4 \Omega^2} \int_0^R r (e_1^2 + f_2^2) dr,$$

so that

$$F_3(x) = \frac{1}{18} \mu^2 a_0^2 \frac{\left(1 + \frac{9}{2} \mu^2\right)}{1 + \frac{1}{2} \mu^2}, \quad \left. \begin{array}{l} (20) \\ (contd.) \end{array} \right\}$$

$$F_5(x) = a_2 \left\{ -\frac{17}{18} \mu^2 a_0 + a_2 \right\}.$$

Substituting for a_0, \bar{a}_1, a_2, b_2 from equations (16) and (18)

$$F_2(x) = \frac{\mu^2 \left(x + \frac{4}{3} \theta_0 \right) \left\{ 3x \left(1 + \frac{1}{6} \mu^2 \right) + \frac{4}{3} \theta_0 \left(1 + \frac{3}{2} \mu^2 \right) \right\}}{\left(1 - \frac{1}{2} \mu^2 \right)^2}, \quad \left. \begin{array}{l} \\ \\ \\ \\ (21) \end{array} \right\}$$

$$F_3(x) = \frac{1}{18} \gamma^2 \mu^2 \frac{\left(1 + \frac{9}{2} \mu^2 \right)}{1 + \frac{1}{2} \mu^2} \left\{ \frac{1}{3} x + \frac{1}{4} \theta_0 (1 + \mu^2) \right\}^2.$$

Writing equation (17) in the form

$$3a_2 - \frac{1}{2} \gamma b_2 = \gamma \mu^2 C,$$

$$\frac{1}{2} \gamma a_2 + 3b_2 = \gamma^2 \mu^2 D,$$

where

$$C = \frac{2}{3} x + \frac{23}{36} \theta_0,$$

$$D = \frac{7}{36} \left(\frac{1}{3} x + \frac{1}{4} \theta_0 \right),$$

it may be verified that

$$(a_2^2 + b_2^2) \left(9 + \frac{1}{4} \gamma^2 \right) = \gamma^2 \mu^4 (C^2 + \gamma^2 D^2).$$

Using this relation to simplify equation (20) it can be shown that

$$\left\{ F_4(x) + F_5(x) \right\} = -\frac{5}{54} \frac{\gamma^2 \mu^4}{9 + \frac{1}{4} \gamma^2} \left\{ x^2 + \frac{37}{24} x \theta_0 + \frac{91}{96} \theta_0^2 \right. \\ \left. + \frac{7}{12} \gamma^2 \left(\frac{1}{3} x + \frac{1}{4} \theta_0 \right)^2 \right\}. \quad \dots \quad (22).$$

This verifies that the right hand side of the equation of zero torque is a homogeneous quadratic in x and θ_0 for the case of straight blades neglecting gravity. To determine the effect on the performance it is necessary to solve this quadratic for x and substitute in equation (14) for the thrust coefficient t . The resulting value of t is then substituted in equation (1) to determine X/Z for given values of $\sigma t/\mu^2 =$ approximate lift coefficient k_x .

5. *The effect of curvature of the blades.*—For straight blades the zero of the flapping angle β has tacitly been taken as the line of the blade, but for curved blades the zero of β has not so far been defined. If the blade is assumed to have the form of a circular arc as in R. & M. 1111, it will be found convenient to take the zero of β to be the tangent line to the arc at the root, not the chord line as in R. & M. 1111. Since χ is defined as the inclination of the tangent to the arc, at radius r , to the zero line it follows that—

$$\chi = -8 \varepsilon r/R,$$

where ε is the camber of the arc, so that χ is proportional to the radius.

The only term in the equation of zero torque which involves the curvature to the order μ^2 (apart from the dynamical effect of the term J_1 on the value of a_0) is e_1 . Equation (10) gives—

$$e_1 = - (a_0 + \chi) \mu R \Omega + b_1 r \Omega + \text{terms of order } \mu^3,$$

and equation (13) gives—

$$b_1 = \frac{4}{3} \mu a_0 + \frac{4}{R^3} \int_0^R \chi r^2 dr + \text{terms of order } \mu^3.$$

Hence, the part of e_1 depending on χ is—

$$- \chi \mu R \Omega + \frac{4 \mu r \Omega}{R^3} \int_0^R \chi r^2 dr$$

and this vanishes identically when χ is proportional to r .

It is considered sufficient in the present analysis to neglect terms beyond μ^2 involving the curvature; to this order it appears that the entire effect of the curvature on the performance is included by taking the tangent to the circular arc at the origin as the zero of β , and by taking account of the effect of the term C in equation (13) for a_0 , so that equation (16) is replaced by—

$$a_0 = \gamma \left\{ \frac{1}{3} x + \frac{1}{4} \theta_0 (1 + \mu^2) \right\} - C,$$

where

$$C = J_1/I_1 + G_1/I_1 \Omega^2.$$

For a circular arc referred to the tangent at the origin as zero line—

$$J_1 = -3 \varepsilon I_1,$$

where ε is the camber; this replaces the relation $J_1 = \varepsilon I_1$ which holds when the chord of the arc is taken as zero line, as in R. & M. 1111. The only change in the equation of zero torque to the order considered is to replace equation (21) for $F_3(x)$ by the equation—

$$F_3(x) = \frac{1}{18} \gamma^2 \mu^2 \frac{1 + \frac{9}{2} \mu^2}{1 + \frac{1}{2} \mu^2} \left\{ \frac{1}{3} x + \frac{1}{4} \theta_0 (1 + \mu^2) - C/\gamma \right\}^2.$$

6. *Effect of a periodic induced velocity due to curvature of the streamlines.*—A periodic induced velocity of the type contemplated in section (10) of R. & M. 1111 contributes to x an additional term which is proportional to $r \cos \Psi'$. Comparison with equation (4) shows that this is equivalent to an additional curvature χ proportional to r ; the argument of the last section shows that this has zero effect on the lift and drag as far as terms of order μ^2 in the equation of zero torque. This is in agreement with the conclusion of R. & M. 1111, section (10).

7. *Actual Calculation.*—Calculations of the flapping angle and of the lift and drag have been carried through in detail on the basis of the above analysis, for the particular case of the standard full scale autogyro for which $\delta = 0.006$, $\sigma = 0.2$, $\theta_0 = 2^\circ$, as follows:—

- (1) Including the terms $F_1(x)$ only in the equation of zero torque; this gives the case of fixed blades.
- (2) Including $F_1(x)$ and $F_2(x)$; this applies to infinitely heavy blades free to flap and is identical with the case treated in Part I.
- (3) Including $F_1(x)$, $F_2(x)$, and $F_3(x)$ for the case of straight blades with $\gamma = 10$; this includes the effect of the "coning" angle for a machine with straight blades having the same moment of inertia as the present full scale machine. The value $\gamma = 10$ is in agreement with the value assumed by Glauert in R. & M. 1111. (The values:— $W_1/W = 0.03$, $W/\pi R^2 = 2$, $\sigma = 0.2$, $R = 17.5$, assumed on page 9 of R. & M. 1111, give $\gamma = 10.0$). It is more likely that this value of γ is an overestimate than an underestimate, since the value of W_1/W for the full scale autogyro described in T. 2155* is 0.0455 and the value of γ derivable from the figures of that report is roughly 6.5. This value should be increased if there is a concentration of the weight of the blade near the hinge, since the moment of inertia I_1 is calculated by putting

* Unpublished.

$\mu_2 = \frac{1}{3}$ in equation (7) of R. & M. 1111, which assumes a constant distribution of weight along the blade. There is evidently some uncertainty as to the true value of γ , and since the effect of the "coning" angle is proportional to γ^2 , it would be of interest to determine I_1 for an actual full scale autogyro by swinging a blade as a pendulum about its hinge, as has been done for the models.

- (4) Includes the same terms as (3), but with the addition of a curvature of the blade in the same direction as in the full scale machine; the blade is assumed to have the form of a circular arc of camber 0.03 as in R. & M. 1111, and the effect is treated by the approximate method of Section 5. The effect of gravity as estimated in R. & M. 1111 is also included.
- (5) The second order terms $F_4(x)$ and $F_5(x)$ are included, for the case of straight blades with $\gamma = 10.0$.

In all the above cases the values of x have been worked out by solving the quadratic equation of zero torque for a series of even values of μ^2 . Equations (14) and (1) then give the values of t , $\sigma t/\mu^2$ (approx. = k_r) and $X/Z = (X/Z)_0 + \frac{1}{2}k_z$. The Table also gives the values of a_0, a_1, b_1, a_2, b_2 for the case of straight blades with $\gamma = 10$.

Curves of $(X/Z)_0$ are plotted against $\sigma t/\mu^2$ in Fig. 3 for the five cases just mentioned. Three curves derivable from R. & M. 1111 are added for comparison:—

- (6) Calculating the value of $(X/Z)_0$ from formula 1 of the present report or formula (c) of R. & M. 1111, Appendix I, but using the values of T_c and λ calculated as in R. & M. 1111. This method takes no account of the effect of flapping or of coning.
- (7) Standard results of R. & M. 1111. These apply to infinitely heavy blades.
- (8) Including the effect of curvature and coning by the methods of R. & M. 1111.

8. *Discussion of results.*—The main conclusions from the curves of Fig. 3 are as follows:—(1) The effect of the coning angle is to decrease x , i.e., to reduce the rotational speed for given values of μ and V and therefore to reduce the efficiency. (2) The additional effect of the existing curvature is in the same direction and is even more important. (3) The effect of the second order terms is in the opposite direction, i.e., to increase the efficiency, but is absolutely negligible for $\gamma = 10$, $\mu = 0.5$, and a fortiori negligible for heavier blades.

It appears from the analysis that the effect of curvature and coning could be reduced practically to zero by giving the blades a suitable curvature in the opposite direction, and the results suggest

that this would improve the efficiency. Before accepting this conclusion as final it is necessary to notice that the reduction of efficiency due to coning is produced by the increase of terms in the equation of zero torque which represent a torque tending to increase rotation. It is possible for example that the existing curvatures of the blades may allow the use of a larger blade angle with safety from risk of the windmill coming to rest, and the results of Part I suggest that this will be an advantage.

In this connection it may be remarked that the freedom of the blades to cone is an additional safeguard against any sudden accidental increase of load, which will have the effect of first increasing the coning angle and will give additional assistance in increasing the rotational speed. An occurrence of this kind was actually observed on one occasion when the machine was taking off, when the coning angle suddenly increased and then immediately recovered.

Finally, it may be worth while to attempt to meet the criticism that in the analysis of Part II a structure is erected on the basis of assumption (2) of Part I which is too heavy for it to bear. In this connection is it worth while to remark that for given values of α and μ the whole of the present analysis of the flapping motion depends only on the lift of the elements, and that the lift is likely to be much less affected than the drag by the stalling of the blade sections. In a future paper an attempt will be made to allow for the effect of stalling on the mean profile drag coefficient in the equation of zero torque and in the equation of energy loss; this further correction will leave the main analysis of Part II unaltered, and will only affect the numerical results by substituting for the constant δ in equations (1) and (15) a quantity which may be expected to be a function of μ and α .

TABLE.

Effect of flapping on the lift and drag of the standard autogyro: $\delta = 0.006$, $\sigma = 0.2$, $\theta_0 = 2^\circ$.
 Values of x , $\sigma t/\mu^2$ approxm. = k_z and $(X/Z)_0 = (X/Z) - \frac{1}{2} k_z$ for even values of μ^2 .

μ^2	μ	0	0.04	0.08	0.12	0.16	0.20	0.24
		0	0.2000	0.2827	0.3463	0.400	0.447	0.490
Fixed blades	x	0.02206	0.0227	0.0233	0.0239	0.0245	0.0249	0.0253
	$\sigma t/\mu^2$		0.354	0.185	0.128	0.100	0.083	0.071
	$(X/Z)_0$		0.119	0.089	0.077	0.069 ₅	0.065	0.062
Heavy blades, free to flap	x	0.02206	0.0182	0.0148	0.0117	0.0087	0.0059	0.0035
	$\sigma t/\mu^2$		0.321	0.152	0.098	0.070	0.055	0.044
	$(X/Z)_0$		0.131	0.108	0.1005	0.098 ₅	0.098	0.100
Effect of coning straight blades $\gamma = 10.0$	x	0.02206	0.0175	0.0134	0.0096	0.0064	0.0032	0.0003
	$\sigma t/\mu^2$		0.315	0.148	0.092	0.066	0.050	0.040
	$(X/Z)_0$		0.1335	0.111	0.106	0.105	0.107	0.110
Ditto, including second order terms.	x	0.02206	0.0176	0.0137	0.0101	0.0070	0.0039	0.0012
	$\sigma t/\mu^2$		0.316	0.149	0.093	0.067	0.051	0.040
	$(X/Z)_0$		0.133 ₅	0.110 ₅	0.105	0.104	0.106	0.108 ₅
Effect of coning and curvature $\gamma = 10$, $C = 0.076$	x	0.02206	0.0165	0.0113	0.0068	0.0026	-0.0012	-0.0049
	$\sigma t/\mu^2$		0.308	0.140	0.085	0.059	+0.044	+0.033
	$(X/Z)_0$		0.136 ₅	0.117 ₅	0.115 ₅	0.118	0.122	0.132

Values of the coefficients in the Fourier expansion of the coning angle in degrees, for $\gamma = 10.0$ and straight blades.

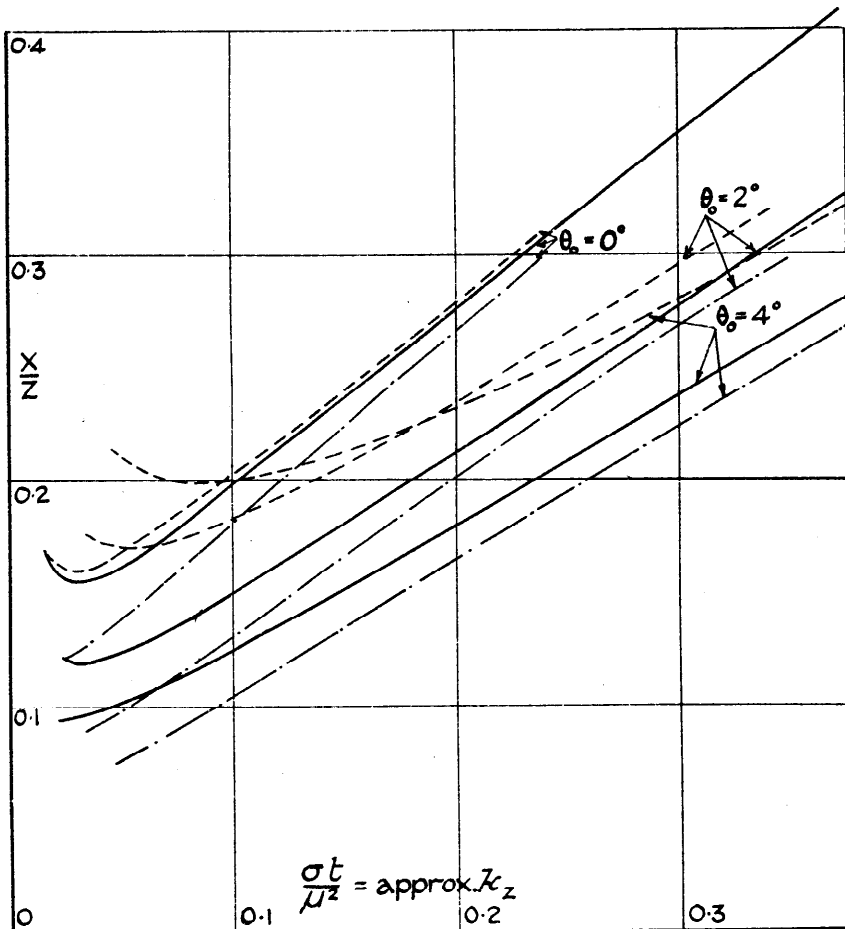
μ^2	0	0.04	0.08	0.12	0.16	0.20	0.24
a_0	9.21	8.35	7.60	6.93	6.33	5.74	5.23
a_1	0	1.48	2.02	2.40	2.69	2.93	3.13
b_1	0	2.16	2.81	3.11	3.24	3.26	3.22
a_2	0	+0.16	+0.29	+0.40	+0.48	+0.54	+0.59
b_2	0	-0.07	-0.12	-0.19	-0.25	-0.32	-0.38

Note.—The values of a_0 , a_1 , b_1 , a_2 , b_2 were obtained by substituting values of x from the previous table for $\gamma = 10$ straight blades, in equation (13); the values of a_2 , b_2 are therefore not in exact agreement with the result of substituting in equation (18).

Lift and drag of an ideal autogyro with heavy blades.

Curves of $\frac{X}{Z}$ plotted against $\frac{\sigma t}{U^2}$ (which is approximately equal to k_z) for $\delta = 0.006$, $\sigma = 0.20$, $\theta_0 = 0^\circ, 2^\circ, 4^\circ$.

- Formulae of present report.
- R.&M.1111, Flapping blades.
- · - · - R.&M.1111, Blade angle varied.



EFFECT OF FLAPPING ON THE
STANDARD AUTOGYRO.

$\delta = 0.006. \quad \sigma = 0.2. \quad \theta_0 = 2^\circ.$

$(x/z)_0 = (x/z) - \frac{1}{2}K_z$

