R. & M. No. 1972
(7514)
A.R.C. Technical Report

MINISTRY OF AIRCRAFT PRODUCTION

AERONAUTICAL RESEARCH COMMITTEE
REPORTS AND MEMORANDA

The Initial Buckling of Slightly Curved Panels Under Combined Shear and Compression

By

D. M. A. LEGGETT, Ph.D.

LONDON: HIS MAJESTY'S STATIONERY OFFICE
Price 3*s.* 0*d.* net

The Initial Buckling of Slightly Curved Panels Under Combined Shear and Compression

By
D. M. A. LEGGETT, Ph.D.

COMMUNICATED BY THE DIRECTOR OF SCIENTIFIC RESEARCH, MINISTRY OF AIRCRAFT PRODUCTION

Reports and Memoranda No. 1972
December, 1943*

Summary.—The initial buckling of flat rectangular panels under combined shear and compression has been investigated theoretically in R. & M. 1965. This report extends the results given there to panels which are long and slightly curved.

On aircraft with laminar flow wing sections, it is desirable that the wing cover should remain smooth up to a factor of $1\frac{1}{4}g$, and to achieve this a possible type of construction is one in which stringers are dispensed with, and the cover is reinforced with closely spaced ribs and stiffeners. These divide the cover into a large number of long and slightly curved panels, and the results given in this report should be of value in estimating the combined shear and compression which such panels can carry without buckling and so developing waviness.

LIST OF CONTENTS

1. *Introduction.*—The initial buckling of flat rectangular panels under combined shear and compression has been investigated theoretically in R. & M. 1965, and the purpose of this report is to extend the results given there to panels which are long and slightly curved.

For flat panels, it is shown in R. & M. 1965[1] that the combination of shear stress q, and compressive stress f, which will just cause buckling are connected by the relation—

$$\left(\frac{q}{q_{cr}}\right)^2 + \left(\frac{f}{f_{cr}}\right) = 1,$$

and for the slightly curved panels considered in this report it is assumed that the same relation is approximately correct.[2] On this basis the problem amounts therefore to finding q_{cr} and f_{cr} for various curvatures and dimensions of panel.

* R.A.E. Report No. S.M.E. 3274, received 6th March, 1944.

The buckling of a short cylinder in torsion, of which a long and slightly curved panel in shear can be regarded as a special case (Fig. 1), has already been investigated theoretically by L. H. Donnell.[3] His treatment, however, involves a number of scarcely justifiable assumptions about boundary conditions, and the more accurate solution developed in this report shows that

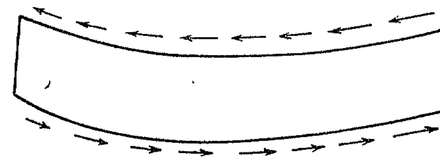

FIG. 1

Donnell's approximate values for the initial buckling stress are appreciably too high. A comparison of the theoretical results found here with the experimental values obtained by E. E. Lundquist[4] shows that for the small curvatures that exist over the major portion of a wing the agreement is good. For larger curvatures, however, owing to the increasing importance of initial irregularities, experimental values are less than the theoretical, and the latter are to be regarded as an ideal upper limit for panels which are perfectly formed and accurately loaded.

The symmetrical buckling of a cylinder under end load, of which a long and slightly curved panel in compression can be regarded as a special case (Fig. 2), has been investigated theoretically by R. V. Southwell, S. Timoshenko[5] and others, and what is done here is to solve the fundamental

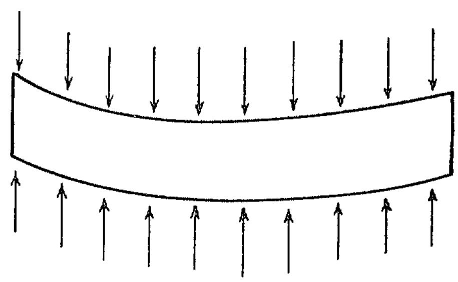

FIG. 2

equations derived by them for alternative boundary conditions. For slightly curved panels there are very few experimental results available. But such evidence as is supplied by tests on curved panels whose straight edges are much longer than their curved ones, suggests that even for the small curvatures considered in this report experimental values of the buckling stress are somewhat less than the theoretical.

2. *Statement of Problem and Method of Solution.*—The problem considered is the initial buckling of a long and slightly curved panel under combined shear and compression. The applied shear is constant round the panel, and the compression is uniformly distributed over the two curved edges (Fig. 3). Owing to the length of the panel the type of support for the two straight sides is unimportant, but for the two curved sides results are worked out on the assumption of clamped or simply supported edge conditions.

The method of obtaining q_{cr} and f_{cr} is explained in the appendices.

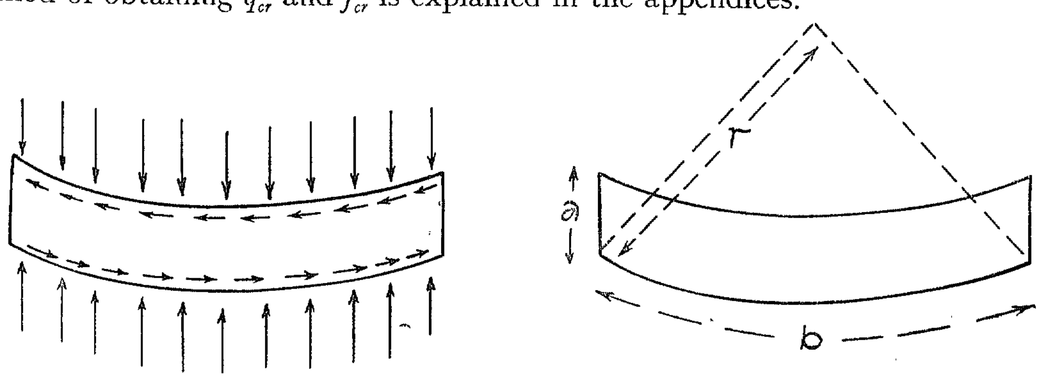

FIG. 3

3. *Description of Results*.—The following notation is used throughout :—

E = Young's modulus
ν = Poisson's ratio (taken as $0 \cdot 25$)
h = semi-thickness of panel
a = width of panel, measured along short straight edge
b = length of panel, measured along long curved edge
r = radius of curvature of panel
K = a non-dimensional constant = $[3(1 - \nu^2)/\pi^4]^{1/2} a^2/rh$
f_{cr} = stress at which panel buckles under compression alone
f_{cro} = value of f_{cr} when panel is flat and the edges are simply supported
f = compression stress at which panel buckles when the shear stress is q
k_{crc} = f_{cr}/f_{cro}
k_c = f/f_{cro}
q_{cr} = stress at which panel buckles under shear alone
q_{cro} = value of q_{cr} when panel is flat, and the edges are simply supported
q = shear stress at which panel buckles when the compression stress is f
k_{crs} = q_{cr}/q_{cro}
k_s = q/q_{cro}

The results are shown graphically in Figs. 4 to 7.

Fig. 4 refers to the case of pure shear, and gives the variation of k_{crs} with K for simply supported or clamped edges. Donnell's results are shown by the broken curves and give values for the buckling stress which are between 10 per cent. and 20 per cent. too high.

Fig. 5 also refers to the pure shear case and shows how the wave length of the buckles varies with K.

Fig. 6 refers to the case of pure compression, and gives the variation of k_{crc} with K when the edges are simply supported or clamped.

Fig. 7 includes the results given in Figs. 4 and 6, and shows, for given K, i.e. for given curvature and size of panel, the various combinations of shear and compression which will just cause the panel to buckle. In Fig. 7 the full and broken curves refer respectively to simply supported and clamped edge conditions.

Table 1 shows the values of K which correspond to typical values of a, r and h.

TABLE 1
Values of K

a	$2h$	r	K	a	$2h$	r	K
6	0·036	100	3·4	9	0·036	100	7·6
		200	1·7			200	3·8
		300	1·1			300	2·5
	0·048	100	2·5		0·048	100	5·7
		200	1·3			200	2·9
		300	0·8			300	1·9
	0·064	100	1·9		0·064	100	4·3
		200	1·0			200	2·2
		300	0·6			300	1·4
	0·080	100	1·5		0·080	100	3·4
		200	0·8			200	1·7
		300	0·5			300	1·1

All dimensions are in inches.

4. *Conclusions*.—This report shows under what combinations of shear and compression a long and slightly curved panel may first be expected to buckle.

In aircraft with laminar flow wing sections, it is desirable that the wing cover should remain smooth up to a factor of $1\frac{1}{4}g$, and to achieve this a possible type of construction is one in which stringers are dispensed with, and the cover is reinforced with closely spaced ribs and stiffeners. These divide the cover into a large number of long and slightly curved panels, and the results in this report should be of value in estimating the combined shear and compression which such panels can carry without buckling and so developing waviness.

LIST OF REFERENCES

No.	*Author*	*Title*
1	H. G. Hopkins and B. V. S. C. Rao ..	"The Initial Buckling of Flat Rectangular Panels under Combined Shear and Compression". R. & M. 1965. 1945.
2	F. J. Bridget, C. C. Jerome, A. B. Vosseller	"Some New Experiments on Buckling of Thin Walled Construction". *Trans. Am. Soc. Mech. Eng.*, Vol. 56, p. 569. 1934.
3	L. H. Donnell	"The Stability of Thin Walled Tubes Under Torsion". N.A.C.A. Report No. 479. 1933.
4	E. E. Lundquist	"Strength Tests on Thin Walled Duralumin Cylinders in Torsion". N.A.C.A. Technical Note No. 427. 1932.
5	S. Timoshenko	"Theory of Elastic Stability". McGraw-Hill, 1936. Chap. 9, p. 81.

APPENDIX I

The Initial Buckling of a Long and Slightly Curved Panel in Shear

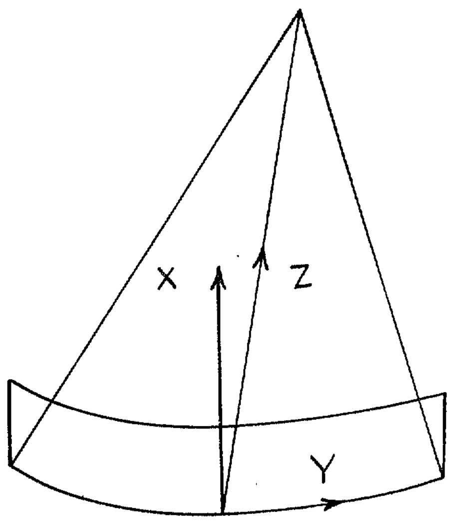

FIG. 8

Additional notation used in Appendix I :—

$$G = \text{shear modulus} = E/2(1 + \nu).$$

The co-ordinate axes OX, OY, OZ are shown in Fig. 8, and are such that OX and OY are the generator and line of curvature through the mid point of one of the curved edges, and OZ is normal to the middle surface. Referred to these axes the edges of the panel are $X = 0,\ a$; $Y = \pm\, b/2$; and the equilibrium displacements u_0, v_0, w_0, are such that

$$u_0 = 0, \quad v_0 = -\frac{q_{cr}\,X}{G}, \quad w_0 = 0.$$

If this configuration is one of neutral equilibrium

$$u_0 + u, \quad v_0 + v, \quad w_0 + w,$$

are also possible displacements, where u, v, w are indefinitely small but not all zero. Substituting each of these sets of displacements in the shell equations obtained by W. R. Dean* for problems of this kind and assuming that a/r and h/a are small and of the same order, the three fundamental stability equations reduce to

$$\frac{\partial}{\partial X}\left[\frac{\partial u}{\partial X} + \nu\left(\frac{\partial v}{\partial Y} - \frac{w}{r}\right)\right] + \frac{(1-\nu)}{2}\frac{\partial}{\partial Y}\left(\frac{\partial v}{\partial X} + \frac{\partial u}{\partial Y}\right) = 0,$$

$$\frac{(1-\nu)}{2}\frac{\partial}{\partial X}\left(\frac{\partial v}{\partial X} + \frac{\partial u}{\partial Y}\right) + \frac{\partial}{\partial X}\left[\left(\frac{\partial v}{\partial Y} - \frac{w}{r}\right) + \nu\frac{\partial u}{\partial X}\right] = 0,$$

$$\frac{h^2}{3}\nabla^4 w + 2(1-\nu^2)\frac{q_{cr}}{E}\frac{\partial^2 w}{\partial X \partial Y} - \frac{1}{r}\left[\left(\frac{\partial V}{\partial Y} - \frac{w}{r}\right) + \nu\frac{\partial u}{\partial X}\right] = 0.$$

---

* *Proc. Roy. Soc.* A, Vol. 107, 1925, p. 734.

Putting these into non-dimensional form by means of the substitutions

$$X = \frac{ax}{\pi},\ Y = \frac{ay}{\pi}$$

we have

$$\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x} + \nu\left(\frac{\partial v}{\partial y} - \frac{aw}{\pi r}\right)\right] + \frac{(1-\nu)}{2}\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = 0,$$

$$\frac{(1-\nu)}{2}\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left[\left(\frac{\partial v}{\partial y} - \frac{aw}{\pi r}\right) + \nu\frac{\partial u}{\partial x}\right] = 0,$$

$$\frac{h^2}{3}\nabla^4 w + 2(1-\nu^2)\left(\frac{a}{\pi}\right)^2\frac{q_{cr}}{E}\frac{\partial^2 w}{\partial x\partial y} - \frac{a^3}{\pi^3 r}\left[\left(\frac{\partial v}{\partial y} - \frac{aw}{\pi r}\right) + \nu\frac{\partial u}{\partial X}\right] = 0.$$

Introducing a stress function f for the purpose of simplifying the analysis, there result the following five equations (only four of which, however, are independent, since equation (4) is obtained by eliminating u and v from (1)–(3)),

$$\frac{(1-\nu)}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -\pi\frac{\partial^2 f}{\partial x\partial y}, \qquad (1)$$

$$\frac{\partial u}{\partial x} + \nu\left(\frac{\partial v}{\partial y} - \frac{aw}{\pi r}\right) = \pi\frac{\partial^2 f}{\partial y^2}, \qquad (2)$$

$$\left(\frac{\partial v}{\partial y} - \frac{aw}{\pi r}\right) + \nu\frac{\partial u}{\partial x} = \pi\frac{\partial^2 f}{\partial x^2}, \qquad (3)$$

$$\nabla^4 f + R\frac{\partial^2 w}{\partial x^2} = 0, \qquad (4)$$

$$\nabla^4 w - P\frac{\partial^2 f}{\partial x^2} + Q\frac{\partial^2 w}{\partial x\partial y} = 0, \qquad (5)$$

where $$P = \frac{3a}{rh^2}\left(\frac{a}{\pi}\right)^2,\quad Q = \frac{6(1-\nu^2)}{h^2}\left(\frac{a}{\pi}\right)^2\frac{q_{cr}}{E},\quad R = \frac{(1-\nu^2)}{ar}\left(\frac{a}{\pi}\right)^2.$$

*Case I. Edges Simply Supported.**—It is now required to find a solution of the equations (1) to (5) which satisfies the boundary conditions

$$\left.\begin{aligned} u = v &= 0,\\ w = \frac{\partial^2 w}{\partial x^2} &= 0,\end{aligned}\right\} \qquad (6)$$

for $x = 0, \pi$; and a solution of w and f is sought for in the form

$$\left.\begin{aligned} w &= w_1\cos my + w_2\sin my,\\ f &= f_1\cos my + f_2\sin my.\end{aligned}\right\} \qquad (7)$$

Here w_1, w_2, f_1, f_2, are functions of x only, and m is real but otherwise unspecified (except in the case of a complete cylinder when $\pi rm/a$ must be an integer).

We express w_s $(s = 1, 2)$ in the form

$$w_s = \sum_{t=1}^{\infty} A_{st}\sin tx\dagger,\ 0 \leqslant x \leqslant \pi$$

* For a more detailed description of the method used, see writer's paper on a similar problem in *Proc. Roy. Soc.* A, Vol. 162, 1937, p. 62.

† By considering the boundary conditions which w_s must satisfy, it can be shown that it is legitimate to differentiate this series term by term four times.

and introduce the complex quantities W, F and A_t, defined by the relations

$$W = w_1 + iw_2, \quad F = f_1 + if_2, \quad A_t = A_{1t} + iA_{2t}.$$

By this means w and f in (7) can be written in the form

$$w = (R)\, We^{-imy}, f = (R)Fe^{-imy}, \qquad (8)$$

where (R) denotes real part.

Substituting for w and f in (4), F is then given by the differential equation

$$\frac{d^4F}{dx^4} - 2m^2\frac{d^2F}{dx^2} + m^4F = R\sum_{t=1}^{\infty} A_t\, t^2 \sin tx,$$

the general solution of which is

$$F = \{A \cosh mx + B \sinh mx\} + x\{C \cosh mx + D \sinh mx\}$$
$$+ R\sum_{t=1}^{\infty} \frac{A_t\, t^2 \sin tx}{(t^2 + m^2)^2} \qquad (9)$$

The next step is to determine the arbitrary constants A, B, C, D in terms of A_t from the first two boundary conditions in (6). Since, however, these involve u and v, it is first necessary to express u and v in terms of F and W. This is done by solving the equations (1), (2) and (3), and the boundary conditions can then be expressed in the form

$$\frac{d^2F}{dx^2} + \nu\, m^2 F = 0,$$

$$\frac{d^3F}{dx^3} - (2+\nu)\, m^2\frac{dF}{dx} + \frac{(1-\nu^2)\, a}{\pi^2 r}\, \frac{dW}{dx} = 0, \qquad (10)$$

for $x = 0, \pi$. Now substituting for F and W in (10), the arbitrary constants A, B, C, D are given by

$$AK = 2RXm\,(1+\nu)\,\{(3-\nu)\sinh m\pi \cosh m\pi + m\pi\,(1+\nu)\}$$
$$- 2RYm\,(1+\nu)\,\{(3-\nu)\sinh m\pi + m\pi\,(1+\nu)\cosh m\pi\},$$

$$BK = RXm\,(1+\nu)\,\{m^2\pi^2\,(1+\nu)^2 - 2\,(3-\nu)\sinh^2 m\pi\}$$
$$- RYm^2\pi\,(1+\nu)^2\,(1-\nu)\sinh m\pi,$$

$$CK = RXm^2\,(1+\nu)^2\,(3-\nu)\sinh^2 m\pi - RYm^3\pi\,(1+\nu)^3\sinh m\pi,$$

$$DK = -RXm^2\,(1+\nu)^2\,\{(3-\nu)\sinh m\pi\cosh m\pi + m\pi\,(1+\nu)\}$$
$$+ RYm^2\,(1+\nu)^2\,\{(3-\nu)\sinh m\pi + m\pi\,(1+\nu)\cosh m\pi\},$$

where

$$K = m^2\,(1+\nu)^2\,\{(3-\nu)^2\sinh^2 m\pi - m\pi^2\,(1+\nu)^2\},$$

$$X = \sum_{t=1}^{\infty} A_t\,\frac{(\nu t^3 - tm^2)}{(t^2+m^2)^2} \equiv \sum_{t=1}^{\infty} A_t K_t,$$

$$Y = \sum_{t=1}^{\infty} (-)^t A_t\,\frac{(\nu t^3 - tm^2)}{(t^2+m^2)^2} \equiv \sum_{t=1}^{\infty} (-)^t A_t K_t.$$

These expressions for A, B, C, D are now substituted in equation (9), with the result that w and f given by (8) will satisfy all the fundamental equations and boundary conditions except (5).

It accordingly remains to substitute for w and f in (5) and equate to zero the coefficient of e^{-imy}. Doing this we have

$$\sum_{t=1}^{\infty} A_t (t^2 + m^2)^2 \sin tx - imQ \sum_{t=1}^{\infty} t\, A_t \cos tx + PR \sum_{t=1}^{\infty} A_t \frac{t^4 \sin tx}{(t^2 + m^2)^2}$$

$$- Pm^2 \{A \cosh mx + B \sinh mx + x\,(C \cosh mx + D \sinh mx)\}$$
$$-2 Pm\,(D \cosh mx + C \sinh mx) = 0, \qquad \ldots \qquad (11)$$

which must be satisfied for all x in $(0, \pi)$. The next step is to express in a Fourier series of sines all those terms in equation (11) which are functions of x, and then to equate to zero the coefficients of the sines. In this way we deduce, after considerable algebraic reduction, the following system of equations

$$(t^2 + m^2)^2 A_t - imQ \sum_{n=1}^{\infty} c_{tn} A_n + \frac{PRt^4}{(t^2 + m^2)^2} A_t$$

$$+ (PRL_t \left\{ \sum_{n=1}^{\infty} K_n A_n + (-)^t \sum_{n=1}^{\infty} (-)^n K_n A_n \right\} = 0. \qquad \ldots \qquad (12)$$

Here $t = 1, 2, \ldots . \infty$, and the c's and L's are given by

$$c_{tn} = \frac{n}{\pi} \left[\frac{1 - \cos(t+n)\pi}{t+n} + \frac{1 - \cos(t-n)\pi}{t-n} \right],$$

$$L_t = \frac{4mK_t T}{\pi\,(1 + \nu)},$$

where
$$TN = \{(3 - \nu) \cosh m\pi \sinh m\pi + m\pi\,(1 + \nu)\}$$
$$- \cos t\pi\, \{(3 - \nu) \sinh m\pi + m\pi\,(1 + \nu) \cosh m\pi\},$$

and
$$N = (3 - \nu)^2 \sinh^2 m\pi - m^2 \pi^2 (1 + \nu)^2.$$

Since the equations (12) are linear in the A's, their only solution is in general that in which all the A's vanish. If however the determinant formed by eliminating the A's itself vanishes this is no longer true, so that the vanishing of this determinant provides the required equation for q_{cr}. The order of this equation being infinite, q_{cr} has an infinite number of roots, one corresponding to each possible form of instability. But since we are only concerned with that form of instability which is most likely to occur in practice, it is only the smallest value of q_{cr} which is of interest.

Introducing Z_t defined by the relation

$$Z_t = (t^2 + m^2)^2 + \frac{PRt^4}{(t^2 + m^2)^2} + 2\,PRK_t L_t,$$

the determinantal equation for the critical shear stress is

$$\begin{vmatrix} Z_1 & -imQc_{12} & 2RPK_3L_1 & -imQc_{14} \\ -imQc_{21} & Z_2 & -imQc_{23} & 2RPK_4L_2 \\ 2RPK_1L_3 & -imQc_{32} & Z_3 & -imQc_{34} \\ -imQc_{41} & 2RPK_2L_4 & -imQc_{43} & Z_4 \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0. \qquad (13)$$

To show that the solution given by equation (13) is not merely a formal one, it is necessary to show that the determinant is convergent. The proof however is on exactly the same lines as that given in the paper already referred to* and so is omitted.

* Loc cit. p. 6.

As it is not possible to solve equation (13) directly, it is necessary to approximate to the solution for Q, i.e. for q_{cr}, by replacing the infinite determinant in that equation by the finite determinant which contains its first n^2 ($n = 2, 3, \ldots$) elements. The results of the successive approximations, and those obtained by Donnell, are given in Table 2.

TABLE 2

K	First Approximation		Second Approximation		Third Approximation		Donnell's Results	
	m^2	k_{crs}	m^2	k_{crs}	m^2	k_{crs}	k_{crs}	Percentage Increase
0	0·59	1·046	0·63	1·000	—	—	1·003	0·3
1	0·75	1·134	0·78	1·071	0·78	1·070	1·144	6·9
2	1·05	1·307	1·1	1·211	—	—	1·331	10·0
3	1·4	1·507	1·45	1·367	—	—	1·515	11·0
4	1·7	1·712	1·8	1·524	1·8	1·521	1·693	11·3
5·5	2·1	2·020	2·3	1·754	—	—	1·947	11·3
7	2·6	2·325	2·7	1·981	2·7	1·976	2·189	10·8
10	—	—	3·6	2·420	3·7	2·400	2·643	10·1
15	—	—	4·9	3·114	5·05	3·069	3·331	8·6

Case II. Edges Clamped.—In this case we again try for a solution of w and f in the form (7), but the boundary conditions to be satisfied are now

$$u = v = 0,$$
$$w = \frac{\partial w}{\partial x} = 0,$$

for $x = 0, \pi$, and it is no longer possible to express w_s as a sine series which can be differentiated term by term four times. Instead, it can be shown by expanding d^4w_s/dx^4 as a sine series and integrating four times, that the corresponding expression for w_s ($s = 1, 2$) is

$$w_s = \sum_{t=1}^{\infty} A_{st} \sin tx + H_s + E_s x + J_s x^2 + G_s x^3. \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

In (14) the sine series is such that it can be differentiated term by term four times, and H_s, E_s, J_s, G_s, are given by

$$H_s = 0, \quad J_s = \frac{1}{\pi} \sum_{t=1}^{\infty} \{2 + (-)^t\} \, tA_{st} \, ,$$

$$E_s = -\sum_{t=1}^{\infty} tA_{st}, \; G_s = -\frac{1}{\pi^2} \sum_{t=1}^{\infty} \{1 + (-)^t\} \, tA_{st}.$$

Introducing E, J and G, defined by the relations

$$E = E_1 + iE_2, \quad J = J_1 + iJ_2, \quad G = G_1 + iG_2,$$

we have, after substituting for w and f in (4), the following differential equation for F

$$\frac{d^4F}{dx^4} - 2m^2\frac{d^2F}{dx^2} + m^4F = R\left\{\sum_{t=1}^{\infty} A_t t^2 \sin tx - 2J - 6\,Gx\right\},$$

which has for its general solution,

$$F = \{A \cosh mx + B \sinh mx\} + x\{\cosh mx + D \sinh mx\}$$
$$+ R\left[\sum_{t=1}^{\infty} \frac{A_t\, t^2 \sin tx}{(t^2+m^2)^2} - \frac{2J + 6\,Gx}{m^4}\right].$$

The general procedure is now very similar to that in Case I. The arbitrary constants A, B, C, D are determined from the boundary conditions, and in place of (11) there ultimately results the equation

$$\sum_{t=1}^{\infty} A_t (t^2 + m^2)^2 \sin tx + \{m^4 (Ex + Jx^2 + Gx^3) - 2m^2 (2J + 6\,Gx)\}$$
$$- imQ \sum_{t=1}^{\infty} tA_t \cos tx - imQ\,(E + 2Jx + 3Gx^2) + PR \sum_{t=1}^{\infty} A_t t^4 \frac{\sin tx}{(t^2 + m^2)^2}$$
$$- Pm^2 \{A \cosh mx + B \sinh mx + x\,(C \cosh mx + D \sinh mx)\}$$
$$- 2Pm\,\{C \sinh mx + D \cosh mx\} = 0.$$

Expressing the left-hand side of this equation as a sine series, and equating to zero the coefficient of $\sin tx$, it follows that

$$(t^2 + m^2)^2 A_t + m^4 \sum_{n=1}^{\infty} e_{tn} A_n - 4\,m^2 \sum_{n=1}^{\infty} j_{tn} A_n - imQ \sum_{n=1}^{\infty} c_{tn} A_n$$
$$- imQ \sum_{n=1}^{\infty} g_{tn} A_n + \frac{PR\, t^4 A_t}{(t^2 + m^2)^2} + PR\left[L_t \left\{\sum_{n=1}^{\infty} A_n (K_n + q_n)\right.\right.$$
$$\left.\left. + (-)^t \sum_{n=1}^{\infty} (-)^n A_n (K_n + q_n)\right\} - (U_t + S_t)\left\{\sum_{n=1}^{\infty} A_n p_n + (-)^t \sum_{n=1}^{\infty} (-)^n A_n p_n\right\}\right] = 0, \quad \ldots \quad (15)$$

where the c's, K's and L's have the same meaning as in Case I, and the e's, j's, g's, p's, q's, U's and S's, are given by

$$e_{tn} = -\frac{4n}{\pi^2 t^3}[2 + (-)^t + (-)^n + 2\,(-)^{t+n}],$$

$$j_{tn} = \frac{2n}{\pi^2 t}[2 + (-)^t + (-)^n + 2\,(-)^{t+n}],$$

$$g_{tn} = -\frac{2n}{\pi t}\{1 - (-)^{t+n}\} + \frac{12n}{\pi^3 t^3}\{1 - (-)^t\}\{1 + (-)^n\},$$

$$p_n = \frac{2\nu n}{\pi m^3}\{2 + (-)^n\},$$

$$q_n = \frac{n}{m^2} + \frac{6\,(2 + \nu)\,n}{\pi^2 m^4}\{1 + (-)^n\},$$

$$U_t = \frac{2\,mt}{N\pi\,(1 + \nu)}\left[\frac{2m\pi\,(1 + \nu)}{(t^2 + m^2)} \cos t\pi \sinh m\pi + \frac{2m^3\pi\,(1 + \nu)^2}{(t^2 + m^2)^2} \cos t\pi \sinh m\pi\right.$$
$$+ \frac{1}{(t^2 + m^2)}\{(3 - \nu)\,(1 - \nu) \sinh^2 m\pi - m^2\pi^2 (1 + \nu)^2\}$$
$$\left. - \frac{2m^2 (1 + \nu)\,(3 - \nu)}{(t^2 + m^2)^2} \sinh^2 m\pi\right],$$

$$S_t = \frac{4\,tm}{\pi N\,(t^2 + m^2)}[(3 - \nu) \sinh^2 m\pi - m\pi\,(1 + \nu) \cos t\pi \sinh m\pi].$$

The equation for q_{cr} is now found from the condition that there exists a non-zero solution of the system of equations (15).

Introducing N_{st} defined by the relations

$$N_{ii} = (i^2 + m^2)^2 + m^4 e_{ii} - 4\,m^2 j_{ii} + \frac{PR\,i^4}{(i^2 + m^2)^2}$$
$$+ 2\,PR\left[L_i\,(K_i + q_i) - p_i\,(U_i + S_i)\right],$$
$$N_{st} = m^4\,e_{st} - 4\,m^2\,j_{st}$$
$$(s \neq t)$$
$$+ PR\left\{1 + (-)^{s+t}\right\}\left[L_s\,(K_t + q_t - p_t\,(U_s + S_s)\right]$$

this gives rise to the determinantal equation

$$\begin{vmatrix} N_{11} & -imQ\,(c_{12}+g_{12}) & N_{13} & -imQ\,(c_{14}+g_{14}) \\ -imQ\,(c_{21}+g_{21}) & N_{22} & -imQ\,(c_{23}+g_{23}) & N_{24} \\ N_{31} & -imQ\,(c_{32}+g_{32}) & N_{33} & -imQ\,(c_{34}+g_{34}) \\ -imQ\,(c_{41}+g_{41}) & N_{42} & -imQ\,(c_{43}+g_{43}) & N_{44} \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0. \quad (16)$$

As in Case I, it is now necessary to consider the convergence of the determinant in (16), but the proof is omitted as it is very similar to that given in the paper referred to.*

The procedure is now the same as in Case I, except that attention is confined to the second approximation, found by taking the first three rows and columns of the determinant. The results, together with those derived by Donnell, are given in Table 3.

TABLE 3

	Second Approximation		Donnell's Results	
K	m^2	k_{crs}	k_{crs}	Percentage Increase
0	1·5	1·664	1·680	0·9
1	1·6	1·689	1·787	5·8
2	1·7	1·753	1·949	11·2
3	1·9	1·846	2·124	15·1
4	2·2	1·952	2·302	18·0
5·5	2·65	2·130	2·560	20·2

* Loc. cit. p. 6.

APPENDIX II

The Initial Buckling of a Long and Slightly Curved Panel in Compression

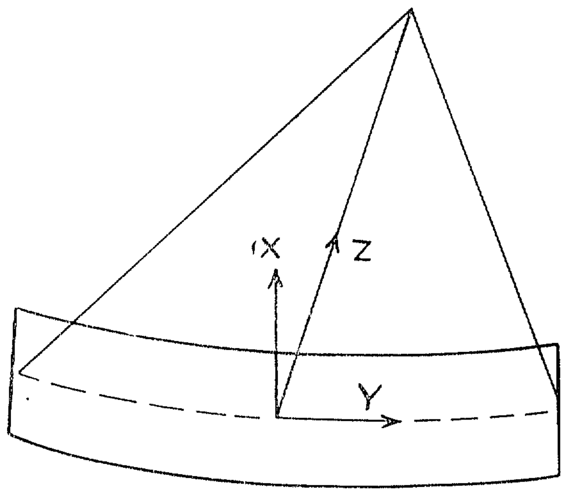

FIG 9.

Additional notation used in Appendix II :—

$$D = \text{flexural rigidity} = 2Eh^3/3\,(1 - \nu^2).$$

The co-ordinate axes OX, OY, OZ are shown in Fig. 9, and the equation of neutral equilibrium, applicable to types of distortion in which there are no displacements parallel to the Y axis, is—

$$D\,\frac{d^4w}{dx^4} + 2hf_{cr}\,\frac{d^2w}{dx^2} + 2\,E\,h\,\frac{w}{r^2} = 0. \qquad \ldots \qquad (17)$$

It remains to find the smallest values of f_{cr} for which there exists a non-zero solution for w satisfying the required boundary conditions.*

Case I. Edges Simply Supported.—For this case the boundary conditions are

$$w = \frac{d^2w}{dx^2} = 0, \text{ for } x = \pm\,\frac{a}{2};$$

$$w \text{ is } A \sin\frac{m\pi x}{a}\ (m \text{ even}) \text{ or } A \cos\frac{m\pi x}{a}\ (m\ odd);$$

and the corresponding values of f_{cr} are given by

$$f_{cr} = \frac{Dm^2\pi^2}{2\,ha^2} + \frac{E\,a^2}{m^2\pi^2r^2}. \qquad \ldots \qquad (18)$$

Case II. Edges Clamped.—The boundary conditions are now

$$w = \frac{dw}{dx} = 0, \text{ for } x = \pm\,\frac{a}{2};$$

and the solution is in this case more complicated.

* See S. Timoshenko, " Theory of Elastic Stability ", First edition, p. 81.

If w is an odd function of x, i.e. if the form of distortion is anti-symmetrical with respect to 0, w is

$$A_s \sin \alpha x + B_s \sin \beta x,$$

where A_s, B_s are constants whose ratio is determined by the boundary conditions,

$$\alpha^2 = \left[f_{cr} + \left\{f_{cr}^2 - 2ED\, h/r^2\right\}^{1/2}\right]/D,$$

$$\beta^2 = \left[f_{cr} - \left\{f_{cr}^2 - 2ED\, h/r^2\right\}^{1/2}\right]/D,$$

and the equation for f_{cr} is

$$\frac{\alpha a}{2} \tan \frac{\beta a}{2} = \frac{\beta a}{2} \tan \frac{\alpha a}{2}. \quad \ldots \quad (19)$$

If w is an even function of x, i.e. if the form of distortion is symmetrical with respect to 0, w is

$$A_c \cos \alpha x + B_c \cos \beta x,$$

where A_c, B_c are constants whose ratio is determined by the boundary conditions, and the equation for f_{cr} is

$$\frac{\alpha a}{2} \tan \frac{\alpha a}{2} = \frac{\beta a}{2} \tan \frac{\beta a}{2}. \quad \ldots \quad (20)$$

Owing to the different types of distortion which are theoretically possible, the equations for f_{cr}, i.e. (18), (19) and (20), are multi-valued, and the two curves given in Fig. 6 are the envelope of all possible solutions when the edges are either clamped or simply supported.

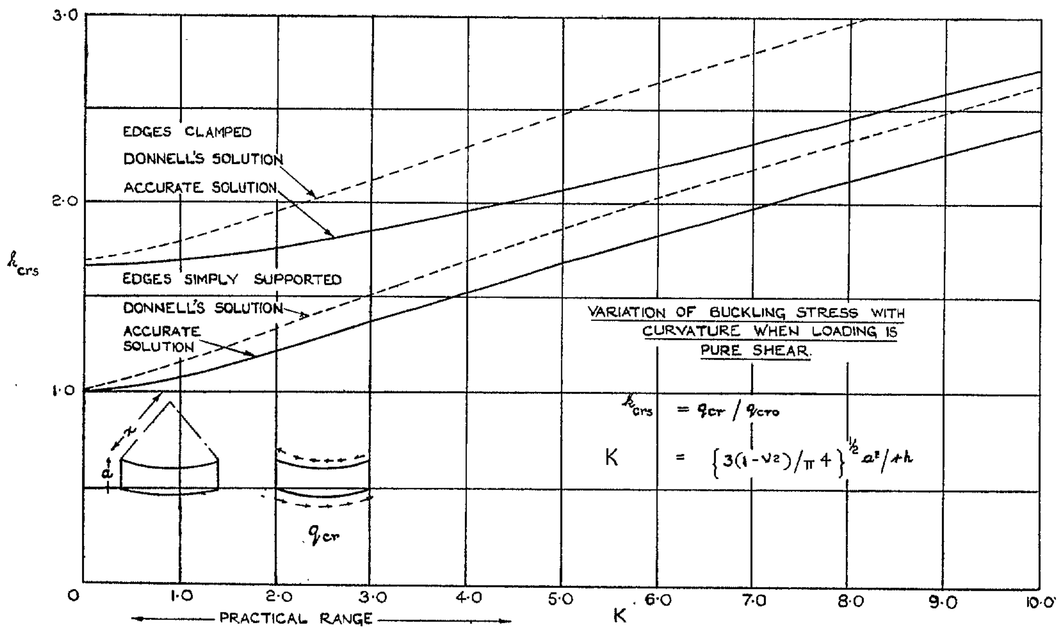
EDGES CLAMPED
DONNELL'S SOLUTION
ACCURATE SOLUTION
EDGES SIMPLY SUPPORTED
DONNELL'S SOLUTION
ACCURATE SOLUTION
VARIATION OF BUCKLING STRESS WITH CURVATURE WHEN LOADING IS PURE SHEAR.
$k_{crs} = q_{cr} / q_{cro}$
$K = \{3(1-\nu^2)/\pi^4\}^{1/2} a^2/rh$
k_{crs}
q_{cr}
K
PRACTICAL RANGE
0
1·0
2·0
3·0
4·0
5·0
6·0
7·0
8·0
9·0
10·0

FIG. 4

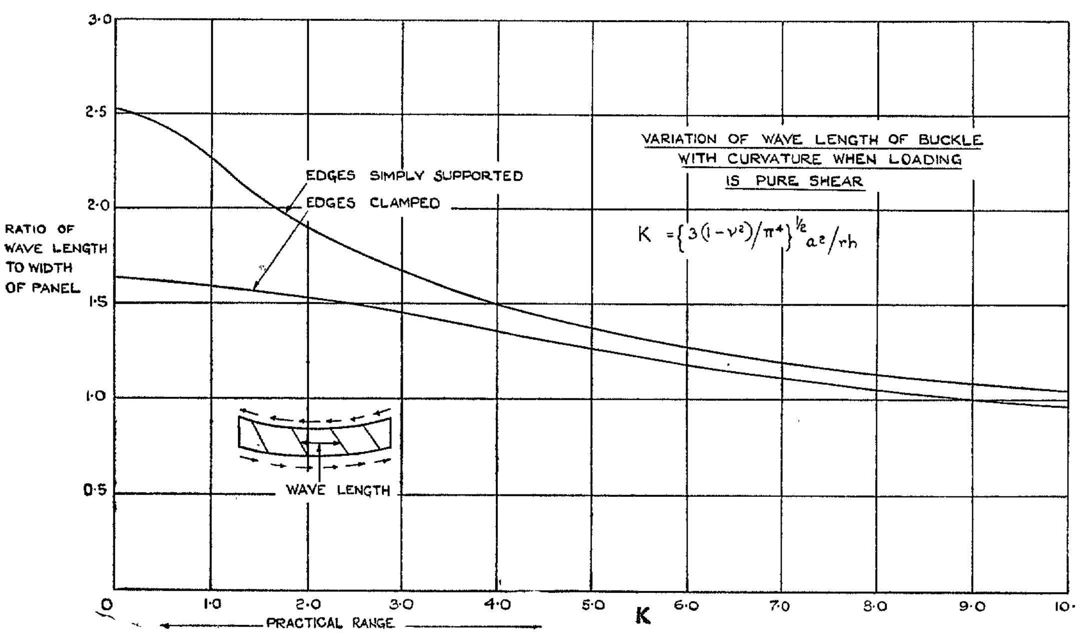
VARIATION OF WAVE LENGTH OF BUCKLE WITH CURVATURE WHEN LOADING IS PURE SHEAR
$K = \{3(1-\nu^2)/\pi^4\}^{1/2} a^2/rh$
EDGES SIMPLY SUPPORTED
EDGES CLAMPED
RATIO OF WAVE LENGTH TO WIDTH OF PANEL
WAVE LENGTH
K
PRACTICAL RANGE
0
1·0
2·0
3·0
4·0
5·0
6·0
7·0
8·0
9·0
10·0
0·5
1·5
2·5

FIG. 5

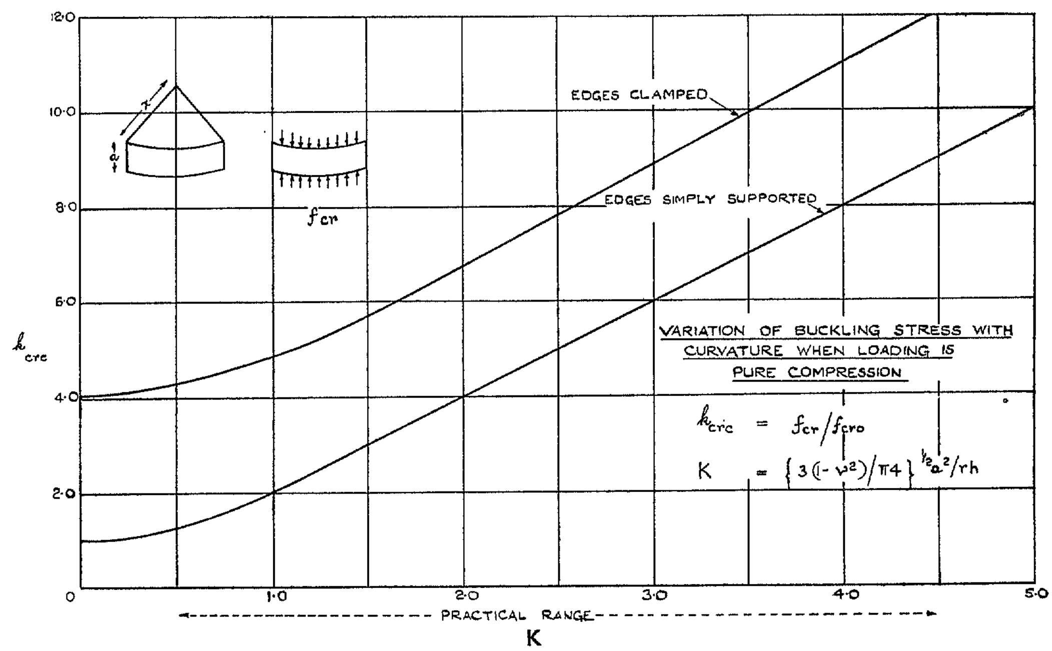
EDGES CLAMPED
EDGES SIMPLY SUPPORTED
VARIATION OF BUCKLING STRESS WITH CURVATURE WHEN LOADING IS PURE COMPRESSION
$k_{crc} = f_{cr}/f_{cro}$
$K = \{3(1-\nu^2)/\pi^4\}^{1/2} a^2/rh$
k_{crc}
PRACTICAL RANGE
K
f_{cr}

FIG. 6

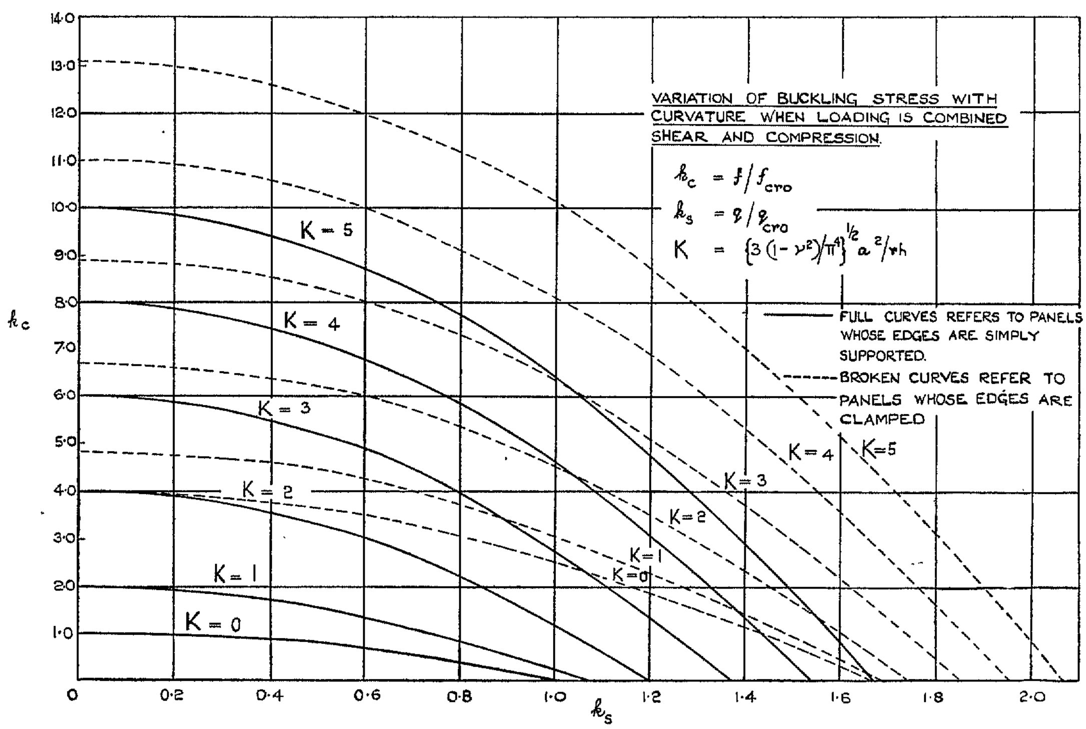
VARIATION OF BUCKLING STRESS WITH CURVATURE WHEN LOADING IS COMBINED SHEAR AND COMPRESSION.
$k_c = f/f_{cro}$
$k_s = q/q_{cro}$
$K = \{3(1-\nu^2)/\pi^4\}^{1/2} a^2/rh$
FULL CURVES REFERS TO PANELS WHOSE EDGES ARE SIMPLY SUPPORTED.
BROKEN CURVES REFER TO PANELS WHOSE EDGES ARE CLAMPED
K = 5
K = 4
K = 3
K = 2
K = 1
K = 0
k_c
k_s

FIG. 7

Publications of the Aeronautical Research Committee

TECHNICAL REPORTS OF THE AERONAUTICAL RESEARCH COMMITTEE—

1934-35 Vol. I. Aerodynamics. 40*s.* (40*s.* 8*d.*)
Vol. II. Seaplanes, Structures, Engines, Materials, etc. 40*s.* (40*s.* 8*d.*)

1935-36 Vol. I. Aerodynamics. 30*s.* (30*s.* 7*d.*)
Vol. II. Structures, Flutter, Engines, Seaplanes, etc. 30*s.* (30*s.* 7*d.*)

1936 Vol. I. Aerodynamics General, Performance, Airscrews, Flutter and Spinning. 40*s.* (40*s.* 9*d.*)
Vol. II. Stability and Control, Structures, Seaplanes, Engines, etc. 50*s.* (50*s.* 10*d.*)

1937 Vol. I. Aerodynamics General, Performance, Airscrews, Flutter and Spinning. 40*s.* (40*s.* 9*d.*)
Vol. II. Stability and Control, Structures, Seaplanes, Engines, etc. 60*s.* (61*s.*)

ANNUAL REPORTS OF THE AERONAUTICAL RESEARCH COMMITTEE—

1933-34 1*s.* 6*d.* (1*s.* 8*d.*)
1934-35 1*s.* 6*d.* (1*s.* 8*d.*)
April 1, 1935 to December 31, 1936. 4*s.* (4*s.* 4*d.*)
1937 2*s.* (2*s.* 2*d.*)
1938 1*s.* 6*d.* (1*s.* 8*d.*)

INDEX TO THE TECHNICAL REPORTS OF THE ADVISORY COMMITTEE ON AERONAUTICS—

1909-1919 Reports and Memoranda No. 1600. 8*s.* (8*s.* 5*d.*)

Prices in brackets include postage.

Obtainable from

His Majesty's Stationery Office

London W.C.2: York House, Kingsway
Edinburgh 2: 13A Castle Street
Manchester 2: 39-41 King Street
Cardiff: 1 St. Andrew's Crescent
Belfast: 80 Chichester Street
or through any bookseller.

S.O. Code No. 23-1972