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The Effect of Various Parameters on Wing-Torsion Aileron-Rotation Flutter

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The Effects of Various Parameters on Wing-Torsion Aileron-Rotation Flutter

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Summary. The effect of certain parameters on binary wing-torsion aileron-rotation flutter are considered in a simplified analysis. These parameters are: aileron circuit stiffness, aileron mass-balance, altitude and structural damping. It is shown that flutter is prevented either by making the aileron natural frequency higher than that of the wing-torsion mode, or by reducing the cross inertia between the modes below a certain critical value. This critical value depends only on three of the coefficients of the Lagrangian flutter equations.

It is concluded that the best means of flutter prevention for a power-controlled aileron is to rely entirely on adequate circuit stiffness, using a fully duplicated power system, and to dispense with mass-balance altogether.

1. *Introduction.* Flutter between the fundamental wing-torsion mode and aileron rotation can be prevented either by adding sufficient mass-balance to the aileron to reduce the cross inertia below a critical value, or by making the aileron natural frequency sufficiently high. When power operation of ailerons was first introduced it was commonly expected that the latter condition would be fulfilled automatically, because of the irreversibility of the control. In practice this is far from being the case. There are two common reasons for this:

(i) When the installation is such that the linkage between the output of the power control and the aileron is long and flexible, for example, when the power unit is mounted in the fuselage.

(ii) When the required aileron frequency is itself high. It has been found in many calculations that this frequency must be greater than that of the wing fundamental torsion mode. The stiffer the wing, therefore, the stiffer the aileron circuit has to be.

It has also been found in many investigations that when the aileron carries some mass-balance, but insufficient to prevent flutter by this means alone, very low flutter speeds can occur if the aileron frequency is just below that of the torsion mode. A typical curve is shown in Fig. 1.

This paper records a general investigation into binary wing-torsion aileron-rotation flutter, from which expressions are derived for the critical values of cross inertia and circuit stiffness to prevent flutter. The effects of altitude and structural damping are also considered. It is concluded that the best way of avoiding flutter of a power-controlled aileron is to design for high circuit stiffness, with duplicated power operation so that mass-balance is unnecessary.

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2. *Analysis.* Denoting the wing-torsion-mode degree of freedom by q_1 and aileron rotation by q_2, the Lagrangian flutter equations can be obtained in their usual form:

$$(-a_{11}\omega^2 + ib_{11}\omega v + c_{11}v^2 + e_{11})q_1 + (-a_{12}\omega^2 + ib_{12}\omega v + c_{12}v^2)q_2 = 0\,, \tag{1}$$

$$(-a_{21}\omega^2 + ib_{21}\omega v + c_{21}v^2)q_1 + (-a_{22}\omega^2 + ib_{22}\omega v + c_{22}v^2 + e_{22})q_2 = 0\,, \tag{2}$$

where ω and v are the non-dimensional frequency and air speed at critical flutter and a, b, c, e are the non-dimensional coefficients relating to inertia, aerodynamic damping, aerodynamic stiffness and structural stiffness respectively. Structural damping has been neglected, for the moment. Now the cross inertia terms, a_{12} and a_{21}, are equal. Also b_{21} and c_{21} are usually small. If these latter two terms are neglected and the equations are scaled so that $a_{11} = a_{22} = 1$, while the symmetry of the inertia matrix is preserved, which may be done without loss of generality, it is shown in the Appendix that the condition that flutter should be prevented at all speeds is:

$$\begin{aligned} &a_{12}^2(4b_{11}b_{22} - b_{12}^2)(c_{22}e_{11} - c_{11}e_{22})^2 + \\ &\quad + 2a_{12}b_{12}(a_{12}c_{12} - b_{11}b_{22})(b_{22}e_{11} - b_{11}e_{22})(c_{22}e_{11} - c_{11}e_{22}) - \\ &\quad - (a_{12}c_{12} - b_{11}b_{22})^2(b_{11}e_{22} + b_{22}e_{11})^2 - \\ &\quad - 4b_{11}b_{22}(a_{12}c_{12} - b_{11}b_{22})(c_{22}e_{11} - c_{11}e_{22})(e_{11} - e_{22}) > 0\,. \end{aligned} \tag{3}$$

It is also shown in the Appendix that if this condition is not satisfied over some range of e_{22} (the aileron circuit stiffness), then as e_{22} is varied, the minimum flutter speed, v_0, is given by:

$$\left.\begin{aligned} &v_0^2 = ke_{11}(1 - c_{11}k)^{-1}\,, \\ \text{where}\qquad &k \simeq a_{12}^2\left(1 - \frac{b_{12}^2}{4b_{11}b_{22}}\right)\left(1 - \frac{a_{12}b_{12}}{2b_{22}}\right)^{-1}(a_{12}c_{12} - b_{11}b_{22})^{-1} \end{aligned}\right\}. \tag{4}$$

The circuit stiffness at which the minimum occurs is given approximately by the equation:

$$e_{22} \simeq e_{11}\left\{\frac{2b_{11}b_{22} - a_{12}b_{12}(b_{11} + b_{22})}{2b_{11}(b_{22} - a_{12}b_{12})}\right\}. \tag{5}$$

We shall now consider under what conditions of circuit stiffness, mass-balance and altitude, flutter is prevented at all speeds, and the flutter speeds which can occur if these conditions do not exist.

3. *The Effect of Parameters.* 3.1. *The Effect of Circuit Stiffness.* It is shown in the Appendix that $(4b_{11}b_{22} - b_{12}^2)$ must be positive; hence the first term in the inequality (3) is always positive. Therefore if

$$a_{12}c_{12} = b_{11}b_{22}\,, \tag{6}$$

the other three terms vanish and the inequality is satisfied for all values of e_{11} and e_{22}. This condition therefore determines a lower bound to the critical cross inertia to prevent flutter. It is discussed further in Section 3.2 below. Hence it is only when $(a_{12}c_{12} - b_{11}b_{22})$ is positive that circuit stiffness is called into play as a flutter preventive. Suppose, then, that $a_{12}c_{12} = \lambda b_{11}b_{22}$, λ being greater than unity. Then the inequality (3) may be rewritten:

$$\begin{aligned} &\left(\frac{\lambda^2}{\lambda - 1}\right)\left(\frac{b_{11}b_{22}}{c_{12}^2}\right)\left(1 - \frac{b_{12}^2}{4b_{11}b_{22}}\right)(c_{22}e_{11} - c_{11}e_{22}) + \\ &\quad + \left(\frac{\lambda b_{12}}{2c_{12}}\right)(b_{22}e_{11} - b_{11}e_{22}) - \left(\frac{\lambda - 1}{4}\right)\frac{(b_{11}e_{22} + b_{22}e_{11})^2}{c_{22}e_{11} - c_{11}e_{22}} + \\ &\quad + e_{22} - e_{11} > 0\,. \end{aligned} \tag{7}$$

The first two terms in this inequality are usually positive, tending to suppress flutter, while the third is negative, tending to promote it. However, it is usually the case that the sum of the three terms is small compared with e_{11} (unless λ is near to unity), so that the sign of the inequality is largely determined by the sign of $(e_{22} - e_{11})$. If it is sufficiently negative, *i.e.*, if the aileron frequency is sufficiently less than the torsion-mode frequency, flutter occurs and if it is sufficiently positive, flutter is prevented. In practice the sum of the first three terms is usually positive, so that the critical aileron frequency to prevent flutter is usually just less than the frequency of the torsion mode. In some cases, however, this sum may be negative and flutter can then occur for a range of values of e_{22} which are greater than e_{11}. The easiest way for this to occur is when λ is large and c_{22} is small, *i.e.*, the aileron has little or no mass-balance but has a fair amount of aerodynamic balance.

It is of interest to note that if $(c_{22}e_{11} - c_{11}e_{22})$ is negative, which would usually imply that the aileron was aerodynamically overbalanced, the sign of the inequality (7) is reversed (since all the terms in (3) have been divided by this factor). In this rather academic case, therefore, flutter is prevented by making the aileron frequency less than the torsion-mode frequency.

It is of interest to examine some curves of flutter speed plotted against aileron stiffness for an actual aeroplane. Table 1 gives the coefficients for a heavy bomber, the cross inertia and aileron stiffness being left as parameters.

TABLE 1

Flutter Coefficients for Heavy Bomber

a	1·000	a_{12}
b	0·052	0·250
c	−0·203	1·089
e	1·000	
a	a_{12}	1·000
b	0·0238	0·418
c	0·0224	0·937
e		e_{22}

It will be seen that the coefficients have been scaled to make $a_{11} = a_{22} = e_{11} = 1 \cdot 0$. Flutter curves for various values of a_{12} are plotted in Fig. 2. It will be seen that, as expected, for all the curves flutter is prevented when $e_{22} > 1 \cdot 0$, *i.e.*, when the aileron frequency is greater than that of the torsion mode.

The minimum flutter speed and the circuit stiffness at which this minimum occurs, as estimated from equations (4) and (5) are also shown in Fig. 2, for three values of a_{12}. It will be seen that fair agreement with the true values is obtained. The discrepancy in minimum flutter speed is due to neglecting b_{21} and c_{21}, while the discrepancy in stiffness is partly due to this and partly due to the other approximations made.

3.2. *The Effect of Mass-balance and Cross Inertia.* In equation (6) we found a lower bound to the value of cross inertia which would prevent flutter at any stiffness. In practice, this value $(b_{11}b_{22}/c_{12})$ will usually be small, so that the inequality (3) will only be satisfied by a small margin. Hence this lower bound should prove to be a fairly close approximation to the true value. For the coefficients given in Table 1 the calculated critical value is 0·0200. It will be seen from Fig. 2 that a very narrow band of flutter still remains when a_{12} has this value. Thus what was thought to be a slightly conservative value has proved to be slightly unconservative, again because b_{21} and c_{21} have been neglected. Even so, the approximation appears to be sufficiently accurate for most purposes.

Next we consider the form of the flutter curves when the mass-balance is insufficient to prevent flutter over some range of stiffness. The effect may be seen either from Fig. 2, or, more conveniently, from Fig. 3. The latter is a cross-plot of Fig. 2, flutter speed being plotted against cross inertia for various values of circuit stiffness. It is clear from the curves that when the circuit stiffness is 0·9 the addition of mass-balance steadily makes the flutter characteristics worse, until the cross inertia is reduced to its critical value of 0·02. The effect is similar when $e_{22} = 0\cdot 6$ but the position starts to improve when the cross inertia falls below 0·14. For zero circuit stiffness the addition of mass-balance is beneficial over most of the practical range of cross inertia.

There is one important point to be remembered in connection with these curves. The cross inertia has been reduced arbitrarily, keeping the direct aileron inertia constant. In practice, for a given aileron structure, reduction in cross inertia can only be brought about by adding mass-balance to the aileron, which increases the direct inertia. In many cases the addition of mass-balance is not very effective in reducing the cross inertia, since the mass-balance weight is usually fairly close to the nodal line of the torsion mode. Thus to achieve a small value of cross inertia a great deal more weight than static balance will usually be required. This weight increases the aileron direct inertia appreciably, thereby reducing the aileron natural frequency (assuming the aileron circuit stiffness is constant). In Fig. 2 this has the effect of pushing the noses of the curves progressively further to the right as the cross inertia is reduced. In Fig. 3 the effect is a little more complicated. Roughly, the curve for a given value of e_{22} is expanded towards the curve for a lower value of e_{22} as the cross inertia is reduced.

The relative merits of mass-balance and circuit stiffness as flutter preventives are discussed in Section 4 below.

3.3. *The Effect of Altitude.* The effect of altitude is normally found by multiplying all the aerodynamic damping terms by $\sqrt{\sigma}$, σ being the relative air density. The flutter speed is then given to the same scale as previously, but in terms of equivalent air speed. Strictly, the aerodynamic inertias should also be factored by σ, but this correction is frequently omitted.

It will be seen from equation (6) that the cross inertia required to prevent flutter is proportional to σ. Hence a system which relies upon mass-balance to prevent flutter may be stable at sea level, but unstable above a certain altitude. On the other hand if no mass-balance is used, reliance being placed upon adequate stiffness, the system is not particularly sensitive to altitude. This is shown in Fig. 4, in which curves are plotted for the same example as previously, but for a height of 40,000 ft. Flutter curves for the two extreme cross inertia values of $a_{12} = 0\cdot 02$ and $0\cdot 50$ are shown, the corresponding curves at sea level being plotted again for comparison. It will be seen that while the curve for the high cross inertia differs little from the sea level case, the band of flutter when $a_{12} = 0\cdot 02$ has been greatly increased by the increase in altitude. Thus any system in which mass-balance is used must be carefully examined to ensure that it is stable at the maximum altitude to which the aircraft will fly.

3.4. *The Effect of Structural Damping.* It has frequently been observed that this type of flutter is sensitive to structural damping, especially to damping in the wing mode. The introduction of structural damping effectively increases the direct aerodynamic damping, by differing amounts at different air speeds. For example, structural damping d_{11} increases the effective aerodynamic damping in mode 1 from b_{11} to $b_{11} + d_{11}/v$, at an air speed v. Now we have shown that the mass-balance required to prevent flutter must be such that the cross inertia is approximately proportional to $b_{11}b_{22}$. It will also be noted that the main effect of the direct aerodynamic dampings on the minimum flutter speed (equation (4)) arises through the product $(b_{11}b_{22})$, the term $(a_{12}b_{12}/2b_{22})$ usually being small compared with unity. Hence the effect of increasing b_{11} by 1 per cent should be the same as increasing b_{22} by 1 per cent, at least as far as the critical mass-balance and minimum flutter speed are concerned, *i.e.*, the effectiveness of structural damping in either mode should be inversely proportional to the direct aerodynamic damping in that mode. Now it is usually the case that $b_{22} >> b_{11}$ (for the example worked out above, for instance, b_{22} was 8 times b_{11}), so that the same amount of structural damping, expressed as a percentage of critical damping, is usually much more effective in the wing mode than it is in the aileron mode. Note also that at height, where $b_{11}b_{22}$ is less than at sea level, damping on the control surface may be particularly ineffective. Fig. 5 shows flutter curves for three combinations of structural damping, for the same example as that shown in Figs. 2 to 4 with $a_{12} = 0 \cdot 1$ and at sea level, as follows:

(1) $d_{11} = d_{22} = 0$

(2) $d_{22} = 0$, $d_{11} = 0 \cdot 025$ ($1 \cdot 25$ per cent critical damping in mode 1)

(3) $d_{11} = 0$, $d_{22} = 0 \cdot 200$ (8 times the previous value of d_{11}).

It will be seen that while cases (2) and (3) are equivalent as far as the minimum flutter speed is concerned, case (2) is much better than case (3) in all other respects. The critical value of the cross inertia to prevent flutter was $0 \cdot 041$ in case (2) and $0 \cdot 042$ in case (3), showing excellent agreement with prediction.

It sometimes happens that an aircraft is built for which neither the aileron stiffness nor the aileron mass-balance can be made quite sufficient, as a result of which calculations show a marginal case of flutter. In such cases the inclusion of structural damping can give a margin of safety, but if the aircraft is to be cleared on this basis (which would only be done as a last resort), it is important to know how much structural damping is present. Since the problem is fairly insensitive to damping in the aileron circuit it is recommended that some effort be devoted to estimating the damping of the wing torsion mode in the ground resonance tests.

4. *Discussion.* We have shown that flutter between the aileron and the wing torsion mode can be prevented either by making the control circuit stiff enough or by making the cross inertia between the two modes low enough. We have also shown that the two methods work against each other; for example, more circuit stiffness would be necessary to prevent flutter for a statically balanced aileron than for an unbalanced aileron, mainly due to the greater inertia of the former. A designer must therefore decide which method he is going to use in any particular case and the advantages and disadvantages of each system will now be considered.

Taking mass-balance first, its main advantage is that no moving parts are involved so that nothing can go wrong. Furthermore, the aircraft is protected no matter what may occur to the stiffness of the power control. Its disadvantage lies in the weight penalty, which may be severe in certain cases, since a great deal more weight than that required for static balance will usually be required. In addition there may be stowage difficulties in getting sufficient mass on to the aileron. A further possible disadvantage associated with mass-balance is that the torsional frequency of the aileron itself is reduced, because of the increased inertia. Thus flutter involving aileron torsion as a degree of freedom may occur. The main advantage of relying on high stiffness is that the weight penalty is avoided, but there are other advantages. For supersonic aircraft the possibility of single-degree-of-freedom flutter occurring at Mach numbers just greater than unity is reduced as the frequency parameter is increased. Hence a high aileron frequency is desirable. A further advantage of this system over mass-balance is that the required stiffness should be known fairly early in the design, whereas the amount of mass-balance required depends on the position of the nodal line of the wing-torsion mode, which is not known accurately until the ground resonance tests are carried out. The only disadvantage of the system is that certain parts of the power unit must be duplicated to guard against failure, *i.e.*, manual reversion cannot be used.

On balance, therefore, the advantage lies with the fully duplicated power control as a flutter preventive, especially for supersonic aircraft. It should be noted that if this method is used some care is necessary in the design to achieve the requisite stiffness. It will usually be found to be essential to mount the operating jack on stiff structure, close to the aileron. The actual servo-mechanism may be remote from the jack, if so desired, provided that screw jacks are used which are operated by high-speed shafts driven by the servo.

5. *Conclusions.* (1) The best method of ensuring that an aircraft is free from wing-torsion aileron-rotation flutter is to make the aileron circuit so stiff that the aileron natural frequency is higher than that of the fundamental wing-torsion mode. A fully duplicated system of power operation should be used, and then no aileron mass-balance is required.

(2) If mass-balance has to be used, care must be taken to ensure that it is sufficient to prevent flutter under all flight conditions, particularly at altitude, which is usually the worst case. It should be noted that mass-balance and circuit stiffness work against each other as far as flutter prevention is concerned, *i.e.*, a small deficiency in stiffness cannot be made up by a sub-critical amount of mass-balance. Either the mass-balance or the stiffness must be sufficient to prevent flutter on its own.

(3) Structural damping in the wing mode has a powerful beneficial effect, and some effort should be devoted to estimating this damping in the ground resonance tests, at least for border-line cases. Structural damping in the control-surface mode is not usually very effective.

LIST OF SYMBOLS

a_{rs}	Non-dimensional coefficient of inertia term in the Lagrangian flutter equations
b_{rs}	Non-dimensional coefficient of aerodynamic damping term in the Lagrangian flutter equations
c_{rs}	Non-dimensional coefficient of aerodynamic stiffness term in the Lagrangian flutter equations
d_{rs}	Non-dimensional coefficient of structural damping term in the Lagrangian flutter equations
e_{rs}	Non-dimensional coefficient of structural stiffness term in the Lagrangian flutter equations
k	A parameter involving some of the above coefficients, defined in equation (4)
v	Non-dimensional air speed
λ	The ratio $\dfrac{a_{12}c_{12}}{b_{11}b_{22}}$
σ	Relative air density
ω	Flutter frequency

APPENDIX

The Approximate Solution of the Lagrangian Flutter Equations

1. *Condition for Flutter to be Prevented at all Speeds.* We are assuming that b_{21} and c_{21} may be neglected and that equations (1) and (2) in the text are scaled so that $a_{11} = a_{22} = 1$, the scaling being such that the symmetry of the inertia matrix is preserved. Then writing $e_{11} + c_{11}v^2 = \omega_1^2$ and $e_{22} + c_{22}v^2 = \omega_2^2$ and cross-multiplying to eliminate q_1 and q_2 we obtain the complex equation:

$$\begin{aligned}(\omega_1^2 - \omega^2 + ib_{11}\omega v)\,(\omega_2^2 - \omega^2 + ib_{22}\omega v) &+ \\ + a_{12}\omega^2(- a_{12}\omega^2 + ib_{12}\omega v + c_{12}v^2) &= 0\,. \end{aligned} \tag{8}$$

Equating the imaginary part of this equation to zero and dividing through by (ωv) gives:

$$b_{11}(\omega_2^2 - \omega^2) + b_{22}(\omega_1^2 - \omega^2) + a_{12}\omega^2 b_{12} = 0\,,$$

whence

$$\omega^2 = \frac{b_{11}\omega_2^2 + b_{22}\omega_1^2}{b_{11} + b_{22} - a_{12}b_{12}}\,. \tag{9}$$

Equating the real part of (8) to zero gives

$$(\omega_1^2 - \omega^2)\,(\omega_2^2 - \omega^2) - b_{11}b_{22}\omega^2 v^2 + a_{12}\omega^2(- a_{12}\omega^2 + c_{12}v^2) = 0\,.$$

Substituting for ω^2 from (9) we have:

$$\begin{aligned}&[b_{11}(\omega_1^2 - \omega_2^2) - a_{12}b_{12}\omega_1^2]\,[b_{22}(\omega_2^2 - \omega_1^2) - a_{12}b_{12}\omega_2^2] - \\ &\quad - a_{12}^2[b_{11}\omega_2^2 + b_{22}\omega_1^2]^2 + \\ &\quad + [a_{12}c_{12} - b_{11}b_{22}]\,[b_{11} + b_{22} - a_{12}b_{12}]\,[b_{11}\omega_2^2 + b_{22}\omega_1^2]v^2 = 0\,. \end{aligned} \tag{10}$$

We can now substitute back for ω_1^2 and ω_2^2, obtaining an equation relating v^2 and e_{22} which is of the form:

$$av^4 + 2he_{22}v^2 + be_{22}^2 + 2gv^2 + 2fe_{22} + c = 0\,, \tag{11}$$

where

$$\begin{aligned}
a &= - b_{11}b_{22}\,(c_{22} - c_{11})^2 + a_{12}b_{12}(c_{22} - c_{11})(b_{11}c_{22} - b_{22}c_{11}) + \\
&\quad + a_{12}^2b_{12}^2c_{11}c_{22} - a_{12}^2(b_{11}c_{22} + b_{22}c_{11})^2 + \\
&\quad + (a_{12}c_{12} - b_{11}b_{22})(b_{11} + b_{22} - a_{12}b_{12})(b_{11}c_{22} + b_{22}c_{11}) \\
b &= - b_{11}b_{22} + a_{12}b_{12}b_{11} - a_{12}^2b_{11}^2 \\
c &= e_{11}^2(- b_{11}b_{22} + a_{12}b_{12}b_{22} - a_{12}^2b_{22}^2) \\
2g &= e_{11}\left\{\begin{aligned}&2b_{11}b_{22}(c_{22} - c_{11}) - a_{12}b_{12}c_{22}(b_{11} + b_{22}) + \\ &\quad + 2a_{12}b_{12}b_{22}c_{11} + a_{12}^2b_{12}^2c_{22} - \\ &\quad - 2a_{12}^2b_{22}(b_{22}c_{11} + b_{11}c_{22}) + \\ &\quad + b_{22}(a_{12}c_{12} - b_{11}b_{22})(b_{11} + b_{22} - a_{12}b_{12})\end{aligned}\right\} \\
2h &= - 2b_{11}b_{22}(c_{22} - c_{11}) - a_{12}b_{12}c_{11}(b_{11} + b_{22}) + \\
&\quad + 2a_{12}b_{12}b_{11}c_{22} + a_{12}^2b_{12}^2c_{11} - 2a_{12}^2b_{11}(b_{22}c_{11} + b_{11}c_{22}) + \\
&\quad + b_{11}(a_{12}c_{12} - b_{11}b_{22})(b_{11} + b_{22} - a_{12}b_{12}) \\
2f &= e_{11}\,[2b_{11}b_{22} - a_{12}b_{12}(b_{11} + b_{22}) + a_{12}^2(b_{12}^2 - 2b_{11}b_{22})]\,.
\end{aligned}$$

Now we require the condition that equation (11) should yield no real values for v. There are two ways in which this can occur; either the values of v^2 obtained can be imaginary or they can be negative. In most cases it is found that the limiting condition of flutter is obtained when v^2 becomes imaginary, *i.e.*, when

$$(he_{22} + g)^2 < abe_{22}^2 + 2afe_{22} + ac\,.$$

At first sight the algebraic values of a, b, c, etc., look so cumbersome that this condition appears too involved to be useful in an algebraic form. Fortunately many of the terms cancel, so that after reduction which includes dividing out by the common factor $(b_{11} + b_{22} - a_{12}b_{12})^2$, we are left with the condition that:

$$\begin{aligned} & a_{12}^2\,(4b_{11}b_{22} - b_{12}^2)(c_{22}e_{11} - c_{11}e_{22})^2 - (a_{12}c_{12} - b_{11}b_{22})^2\,(b_{11}e_{22} + b_{22}e_{11})^2 + \\ & \quad + 2a_{12}b_{12}(a_{12}c_{12} - b_{11}b_{22})(b_{22}e_{11} - b_{11}e_{22})(c_{22}e_{11} - c_{11}e_{22}) - \\ & \quad - 4b_{11}b_{22}(a_{12}c_{12} - b_{11}b_{22})(c_{22}e_{11} - c_{11}e_{22})(e_{11} - e_{22}) > 0\,. \end{aligned} \tag{12}$$

The significance of this inequality is discussed in the text.

2. *Minimum Flutter Speed.* If the aileron mass-balance is such that flutter occurs for a range of aileron circuit stiffness, the magnitude of the minimum flutter speed and the circuit stiffness at which it occurs are of interest. We shall therefore derive approximations to these two quantities.

Referring to equation (10) we substitute:

$$\omega_2^2 = \omega_1^2(1 + x)\,. \tag{13}$$

Equation (10) then becomes:

$$\begin{aligned} & b_{11}x^2(a_{12}^2b_{11} - a_{12}b_{12} + b_{22}) + \\ & \quad + x[2a_{12}^2b_{11}(b_{11} + b_{22}) - a_{12}^2b_{12}^2 - a_{12}b_{12}(b_{11} - b_{22})] + \\ & \quad + a_{12}^2[(b_{11} + b_{22})^2 - b_{12}^2] \\ & \quad = [b_{11} + b_{22} - a_{12}b_{12}][b_{11} + b_{22} + xb_{11}][a_{12}c_{12} - b_{11}b_{22}]\,\frac{v^2}{\omega_1^2}\,. \end{aligned} \tag{14}$$

At the minimum flutter speed, v_0, say, $dv/de_{22} = 0$.
Therefore

$$\left[\frac{dv}{dx}\right]_{v=v_0} = \frac{dv}{de_{22}}\,\frac{de_{22}}{dx} = 0\,.$$

Now equation (14) is of the form:

$$p_0x^2 + p_1x + p_2 = (q_1x + q_2)\,v^2\,. \tag{15}$$

Differentiating with respect to x, putting $dv/dx = 0$ gives:

$$2p_0x + p_1 = q_1v_0^2\,. \tag{16}$$

Substituting this value for x in (14) gives the turning values for v^2 as the roots of the equation:

$$q_1^2v_0^4 + 2(2p_0q_2 - p_1q_1)v_0^2 + p_1^2 - 4p_0p_2 = 0\,. \tag{17}$$

Therefore

$$q_1^2v_0^2 = (p_1q_1 - 2p_0q_2) \pm (4p_0^2q_2^2 - 4p_0p_1q_1q_2 + 4p_0p_2q_1^2)^{1/2}\,. \tag{18}$$

There is some ambiguity regarding the sign of the square root to be taken. By differentiating (15) twice with respect to x, and then putting $dv/dx = 0$ we find that

$$\frac{d^2(v^2)}{dx^2} = \frac{2p_0}{q_1 x + q_2} \qquad \text{at the turning point.}$$

Substituting for x from (16) gives:

$$\frac{d^2(v^2)}{dx^2} = \frac{4p_0^2}{q_1^2 v_0^2 - p_1 q_1 + 2p_0 q_2}.$$

Substituting for v_0^2 from (18) gives:

$$\frac{d^2(v^2)}{dx^2} = \pm\, 4p_0^2(4p_0^2 q_2^2 - 4p_0 p_1 q_1 q_2 + 4p_0 p_2 q_1^2)^{-1/2}.$$

Hence the sign of $d^2(v^2)/dx^2$ is the same as the sign of the square root. Hence for a minimum in v^2 (and hence for a minimum in v) the positive sign must be taken.

On substituting for p_0, p_1, etc., the required value of v_0 is found to be given by

$$\frac{v_0^2}{\omega_1^2} = -\frac{(2b_{22} - a_{12}b_{12}) + 2(a_{12}^2 b_{11} b_{22} - a_{12} b_{12} b_{22} + b_{22}^2)^{1/2}}{b_{11}(a_{12}c_{12} - b_{11}b_{22})}.$$

Assuming that a_{12} is small, the square root may be expanded as a power series in a_{12}. Neglecting terms in this expansion of order higher than a_{12}^2 we find that

$$\left.\begin{aligned} \frac{v_0^2}{\omega_1^2} &\simeq a_{12}^2\left(1 - \frac{b_{12}^2}{4b_{11}b_{22}}\right)\left(1 - \frac{a_{12}b_{12}}{2b_{22}}\right)^{-1}(a_{12}c_{12} - b_{11}b_{22})^{-1} \\ &\simeq k^2, \qquad \text{say} \end{aligned}\right\}. \tag{19}$$

Substituting for $\omega_1^2 = e_{11} + c_{11}v^2$ then gives:

$$v_0^2 = \frac{e_{11}k^2}{1 - k^2 c_{11}}. \tag{20}$$

It will be seen that (19) contains the term $[1 - \{b_{12}^2/(4b_{11}b_{22})\}]$. If this term is zero or negative, then flutter can occur at zero air-speed, *i.e.*, at an infinite frequency parameter. It is known from experience that this cannot happen, so that the derivatives must be such that $4b_{11}b_{22} > b_{12}^2$. This fact is made use of in Section 3.1 of the text.

3. *Circuit Stiffness at which Minimum Flutter Speed Occurs.* The circuit stiffness at which the minimum flutter speed occurs is obtained by substituting for v_0^2 in equation (16).

We first note that

$$\begin{aligned} \frac{q_1}{2p_0} &= \frac{(b_{11} + b_{22} - a_{12}b_{12})}{2(a_{12}^2 b_{11} + b_{22} - a_{12}b_{12})}\,\frac{(a_{12}c_{12} - b_{11}b_{22})}{\omega_1^2} \\ &\simeq \frac{a_{12}c_{12} - b_{11}b_{22}}{2\omega_1^2}, \end{aligned}$$

since b_{11} is small compared with b_{22}.

Therefore

$$\frac{q_1 v_0^2}{2p_0} \simeq \frac{a_{12}^2}{2}\left(1 - \frac{b_{12}^2}{4b_{11}b_{22}}\right)^{-1}\left(1 - \frac{a_{12}b_{12}}{2b_{22}}\right)^{-1}.$$

Also

$$-\frac{p_1}{2p_0} = [-a_{12}b_{12}(b_{22} - b_{11}) + a_{12}^2(b_{12}^2 - 2b_{11}b_{22} - 2b_{11}^2)] \times$$
$$\times\ [2b_{11}(b_{22} - a_{12}b_{12} + a_{12}^2b_{11})]^{-1}.$$

For a first approximation we may neglect a_{12}^2.

Therefore

$$x_0 = \frac{q_1v_0^2 - p_1}{2p_0} \simeq -\frac{a_{12}b_{12}(b_{22} - b_{11})}{2b_{11}(b_{22} - a_{12}b_{12})},$$

x_0 being the value of x when the flutter speed is a minimum.

From (13)

$$1 + x = \frac{\omega_2^2}{\omega_1^2} = \frac{e_{22} + c_{22}v^2}{e_{11} + c_{11}v^2}.$$

Therefore at the minimum flutter speed

$$e_{22} = (e_{11} + c_{11}v_0^2)(1 + x_0) - c_{22}v_0^2. \qquad (21)$$

Since we are neglecting terms of the order of a_{12}^2 we may neglect v_0^2 in (21).

Therefore

$$e_{22} \simeq (1 + x_0)e_{11}$$
$$\simeq e_{11}\left[1 - \frac{a_{12}b_{12}(b_{22} - b_{11})}{2b_{11}(b_{22} - a_{12}b_{12})}\right]$$
$$\simeq e_{11}\left[\frac{2b_{11}b_{22} - a_{12}b_{12}(b_{11} + b_{22})}{2b_{11}(b_{22} - a_{12}b_{12})}\right].$$

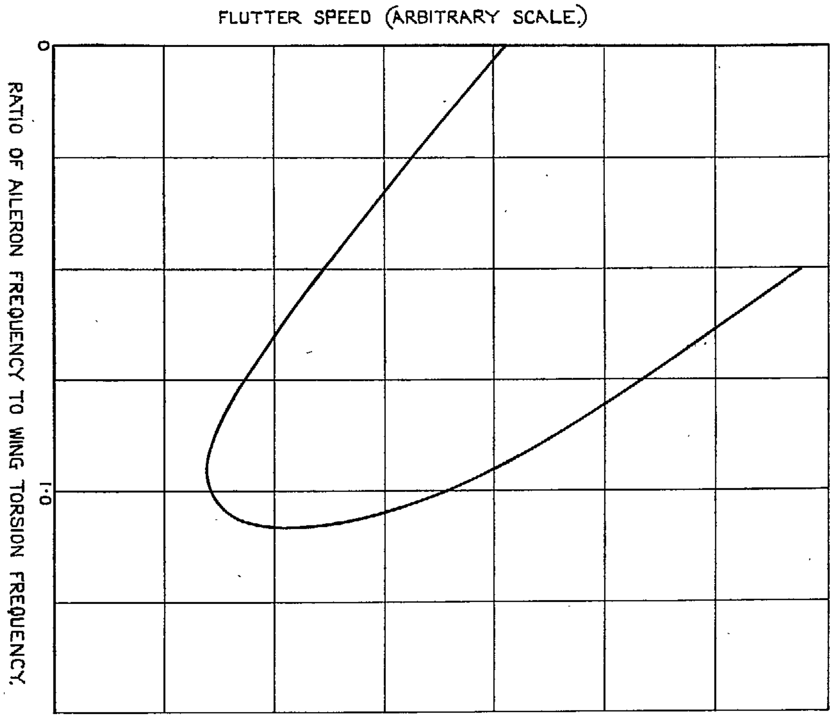

FIG. 1. Typical curve of flutter speed as a function of aileron frequency (Mass-balanced aileron).

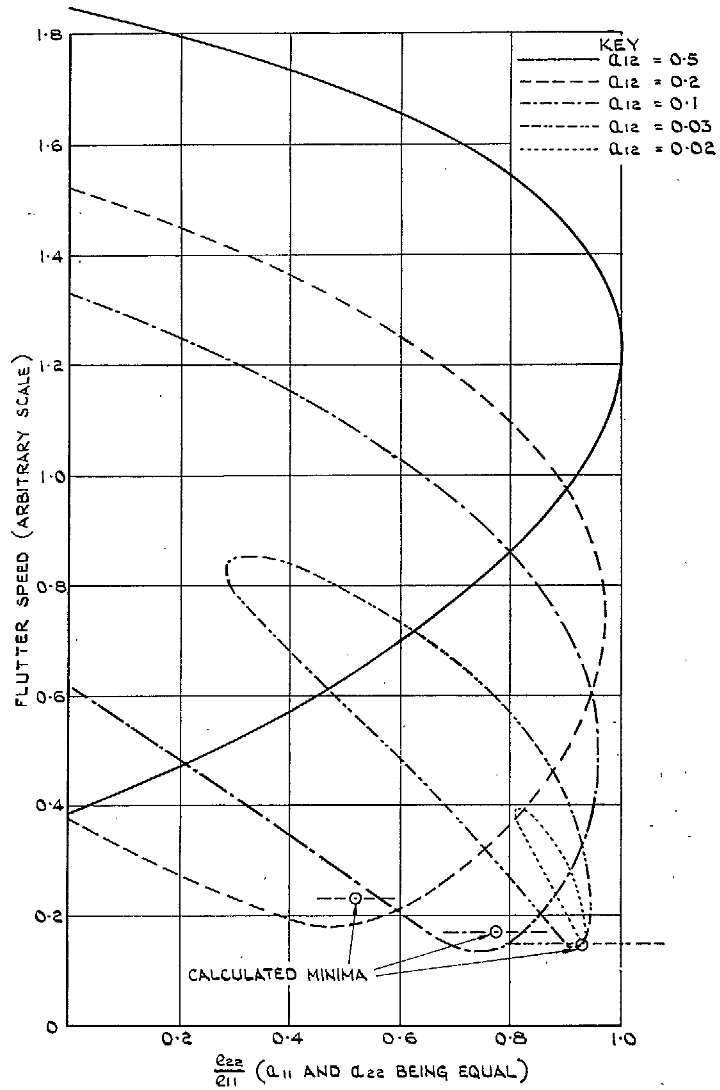

FIG. 2. Flutter speed as a function of aileron stiffness (e_{22}) for various cross inertias.

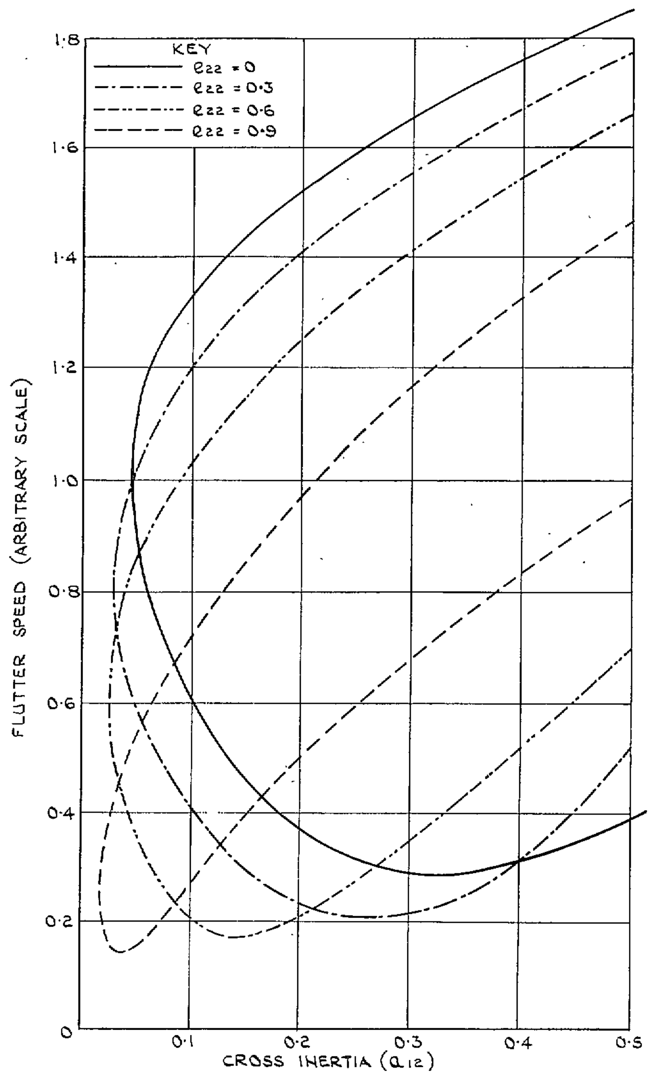

FIG. 3. Variation of flutter speed with cross inertia, for various values of aileron circuit stiffness.

(78682) Wt. 54/8210 K.5 11/60 Hw.

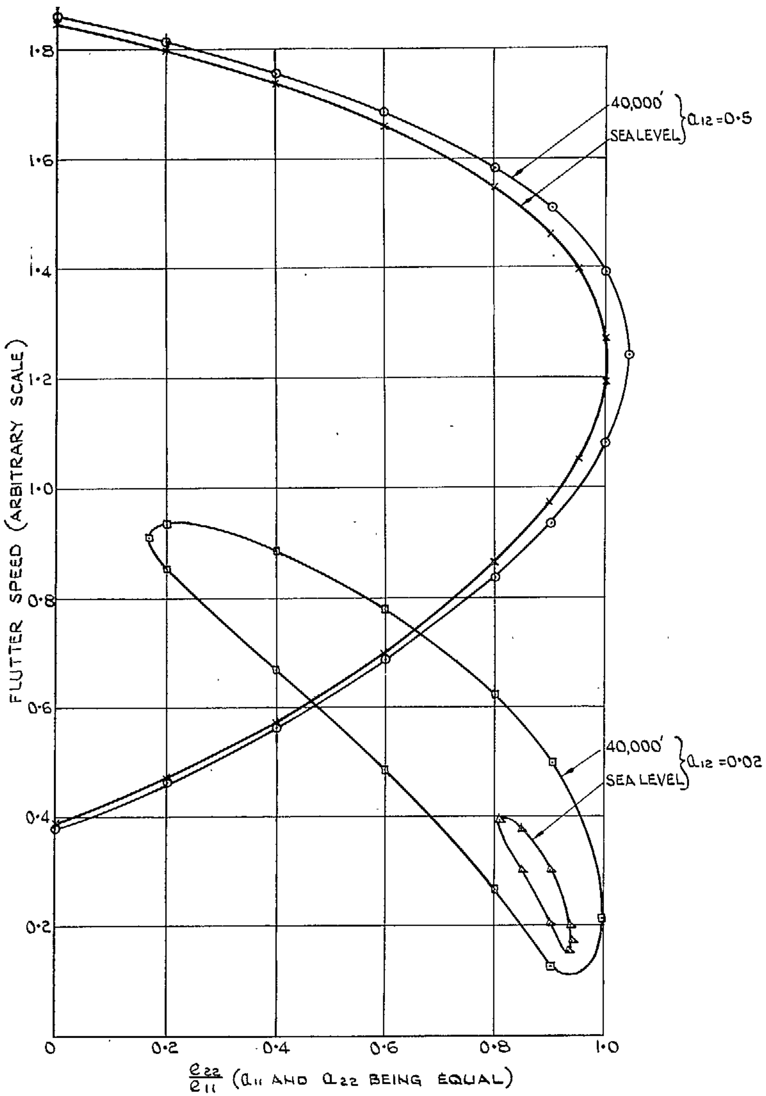

FIG. 4. The effect of altitude on the flutter speed (equivalent airspeed).

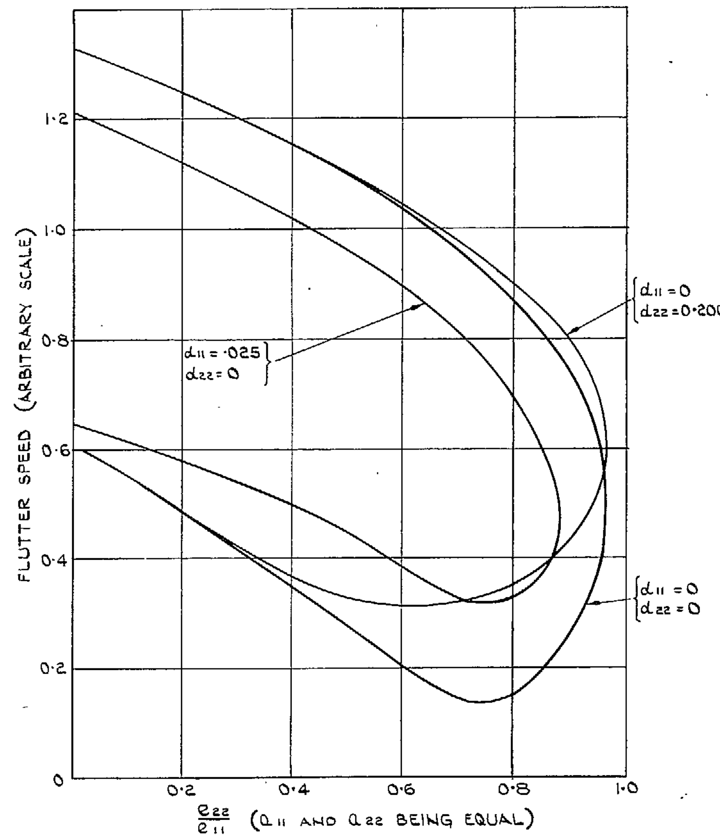

FIG. 5. The effect of structural damping in either mode: $a_{12} = 0 \cdot 1$.

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